A NOTE ON GROUP SCHEMES

1. Preface

When deciding on definitions, I maintain a simple philosophy: whenever possible, a definition should be the "obvious" extension of a familiar definition to a new setting; failing that, it should rely on Yoneda's lemma (which embeds a category $\mathcal{C}$ into the larger category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$, and hence allows one to retreat to the world of sets). In cases where both options are available, I generally endeavor to prove the equivalence of the two choices. It is comforting that definitions thus obtained usually agree with standard definitions found elsewhere, as one would expect for any "good" definition.

2. Group Schemes

We work throughout over a fixed a base scheme S. Let $\mathbf{Sch}_S$ denote the category of schemes over S, whose objects are morphisms $s : T \to S$ and whose morphisms are commutative diagrams

$$\begin{array}{ccc} T & \xrightarrow{\;\;f\;\;} & X \\ & {\scriptstyle s}\searrow \quad \swarrow{\scriptstyle t} & \\ & S & \end{array} .$$

When working with an object $s : T \to S$, we will often suppress the structural morphism s and simply refer to the S-scheme T. Given S-schemes T and X, we denote by $X(T)$ the set $\mathrm{Mor}_S(T, X)$, and refer to elements of $X(T)$ as *T-valued points* of X.

2.1. Definition (cf. [Ma, §III.6], [Mu, Def. 0.1]**).** A *group S-scheme* (also known as a *group object in* $\mathbf{Sch}_S$) is the data of an S-scheme G and a triple of S-morphisms $\mu : G \times_S G \to G$, $e : S \to G$, and $i : G \to G$, making the following diagrams commute:

(i) (Associativity)

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{1\times\mu} & G \times_S G \\ {\scriptstyle \mu\times 1}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times_S G & \xrightarrow{\quad\mu\quad} & G. \end{array}$$

(ii) (Identity)

$$\begin{array}{ccccccccc} G & \xrightarrow{\sim} & S \times_S G & \xrightarrow{e\times 1} & G \times_S G & \xleftarrow{1\times e} & G \times_S S & \xleftarrow{\sim} & G \\ & {\scriptstyle 1}\searrow & & & \downarrow{\scriptstyle \mu} & & & \swarrow{\scriptstyle 1} & \\ & & & & G. & & & & \end{array}$$

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(iii) (Inverse)

$$\begin{array}{ccccccccc} G & \xrightarrow{\Delta} & G \times_S G & \xrightarrow{i\times 1} & G \times_S G & \xleftarrow{1\times i} & G \times_S G & \xleftarrow{\Delta} & G \\ \downarrow & & & & \downarrow{\scriptstyle \mu} & & & & \downarrow \\ S & & \xrightarrow{\quad e \quad} & & G & & \xleftarrow{\quad e \quad} & & S. \end{array}$$

We refer to the morphisms μ, e, and i as the *multiplication, identity,* and *inverse* in G, respectively. Given an S-scheme G, we call a triple (μ, e, i) of S-morphisms as above the *data of a group structure* on G.

2.2. PROPOSITION (CF. [Ma, Prop. 1,§III.6]). — *Given an S-scheme G, the data of a group structure on G is equivalent to a factorization of the functor* $\mathrm{Mor}_S(\cdot, G) : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ *through the forgetful functor* $\mathbf{Grp} \to \mathbf{Set}$.

Proof. Given the data (μ, e, i) of a group structure on G, observe that for every S-scheme T, applying the functor $\mathrm{Mor}_S(T, \cdot)$ to the S-morphisms $\mu : G \times_S G \to G$, $e : S \to G$, and $i : G \to G$ induces set maps $\mu_T : G(T) \times G(T) \cong (G \times_S G)(T) \to G(T)$, $e_T : \{\cdot\} \cong S(T) \to G(T)$, and $i_T : G(T) \to G(T)$. Moreover, applying $\mathrm{Mor}_S(T, \cdot)$ to the commutative diagrams (i)–(iii) exactly gives the commutative diagrams for μ_T, e_T, i_T ensuring they define a group structure on $G(T) = \mathrm{Mor}_S(T, G)$.

Now suppose $f : T_1 \to T_2$ is an S-morphism, and let $f^* : G(T_2) \to G(T_1)$ denote the induced map of sets. From the commutative diagram

$$\begin{array}{ccc} \{\cdot\} \cong \mathrm{Mor}_S(T_2, S) & \xrightarrow{\circ f} & \mathrm{Mor}_S(T_1, S) \cong \{\cdot\} \\ {\scriptstyle e_{T_2}} \downarrow {\scriptstyle e\,\circ} & & {\scriptstyle e\,\circ} \downarrow {\scriptstyle e_{T_2}} \\ G(T_2) = \mathrm{Mor}_S(T_2, G) & \xrightarrow[\circ f]{} & \mathrm{Mor}_S(T_1, G) = G(T_2), \end{array}$$

we have that $f^*(e_{T_2}) = e_{T_1}$. Similarly, from the commutative diagram

$$\begin{array}{ccc} G(T_2) \times G(T_2) \cong \mathrm{Mor}_S(T_2, G \times_S G) & \xrightarrow{\circ f} & \mathrm{Mor}_S(T_1, G \times_S G) \cong G(T_1) \times G(T_1) \\ {\scriptstyle \mu_{T_2}} \downarrow {\scriptstyle \mu\,\circ} & & {\scriptstyle \mu\,\circ} \downarrow {\scriptstyle \mu_{T_1}} \\ G(T_2) = \mathrm{Mor}_S(T_2, G) & \xrightarrow{\quad \circ f \quad} & \mathrm{Mor}_S(T_1, G) = G(T_1), \end{array}$$

we have that f^* is compatible with the group operations. Thus, f^* is indeed a group homomorphism, and we have a factorization of $\mathrm{Mor}_S(\cdot, G) : \mathbf{Sch}_S \to \mathbf{Set}$ through the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$.

Conversely, suppose the functor $\mathrm{Mor}_S(\cdot, G)$ factors through the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$. Then the sets $\mathrm{Mor}_S(T, G)$ are groups functorially in T, and hence have "multiplication" maps $\mu_T : \mathrm{Mor}_S(T, G) \times \mathrm{Mor}_S(T, G) \cong \mathrm{Mor}_S(T, G \times_S G) \to \mathrm{Mor}_S(T, G)$, "identity" maps $i_T : \{\cdot\} \cong \mathrm{Mor}_S(T, S) \to \mathrm{Mor}_S(T, G)$, and "inverse" maps $i_T : \mathrm{Mor}_S(T, G) \to \mathrm{Mor}_S(T, G)$, satisfying the usual three commutative diagrams. The functoriality in T ensures these define natural transformations $\tilde{\mu} : \mathrm{Mor}_S(\cdot, G \times_S G) \to \mathrm{Mor}_S(\cdot, G)$, $\tilde{e}$:

$\mathrm{Mor}_S(\cdot, S) \to \mathrm{Mor}_S(\cdot, G)$, and $\tilde{i} : \mathrm{Mor}_S(\cdot, G) \to \mathrm{Mor}_S(\cdot, G)$, satisfying the same three commutative diagrams. By Yoneda's lemma, this information is equivalent to S-morphisms $\mu : G \times_S G \to G$, $e : S \to G$, and $i : G \to G$ satisfying the usual three commutative diagrams; i.e., a triple (μ, e, i) defining a group S-scheme structure on G.

It is clear that these two constructions (from the data of a group structure on G to a factorization of $\mathrm{Mor}_S(\cdot, G)$ through **Grp**, and conversely) are mutually inverse. □

2.3. Remark. For completeness, note that for an S-scheme $s : T \to S$, the product in $G(T)$ is given by $\mu_T(\cdot, \cdot) = \mu \circ < \cdot, \cdot >$ (where $< \cdot, \cdot >: G(T) \times G(T) \tilde{\to} (G \times_S G)(T)$), the inverse is given by $i_T(\cdot) = i \circ \cdot$, and the identity is given by $e_T = e \circ s$. Also note that for the group $G(S)$ we may equivalently define the group operation μ_S using the isomorphism $S \cong S \times_S S$; i.e., via the composition

$$G(S) \times G(S) \xrightarrow{\cdot \times \cdot} (G \times_S G)(S \times_S S) \xrightarrow{\sim} (G \times_S G)(S) \xrightarrow{\mu \circ} G(S).$$

2.4. Remark. As a consequence of this proposition, many of the basic results about groups automatically extend to group schemes. For example, we may replace the requirement that the "identity" and "inverse" diagrams commute with the seemingly weaker requirement that the "left identity" and "left inverse" diagrams commute. Indeed, if $e : S \to G$ satisfies the natural "left identity" axiom, say, then e_T is a left identity of the group $G(T)$ for every S-scheme T. By basic group theory, e_T is then automatically also a right identity for $G(T)$. As this holds naturally in T, Yoneda's lemma guarantees that e satisfies the corresponding "right identity" diagram. A similar argument holds for the case of the "left inverse", as well as for the fact that the "(left) identity" morphism $e : S \to G$ is uniquely determined by: the morphism μ and the requirement that μ be part of the data defining a group structure on G (requiring the commutativity of the "left identity" diagram rather than the full "identity" diagram).

We can now define a category $\mathbf{GrpSch}_S$ whose objects are group S-schemes and whose morphisms are those S-morphisms compatible with the group structures, i.e., making diagrams such as the following commute:

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle e_G}\uparrow & \nearrow{\scriptstyle e_H} & \\ S & & \end{array} \quad , \qquad \begin{array}{ccc} G \times_S G & \xrightarrow{f \times f} & H \times_S H \\ {\scriptstyle \mu_G}\downarrow & & \downarrow{\scriptstyle \mu_H} \\ G & \xrightarrow[f]{} & H. \end{array}$$

Observe that we have a forgetful morphism $\mathbf{GrpSch}_S \to \mathbf{Sch}_S$, which remembers an S-scheme G but forgets the data (μ, e, i) of the group structure.

2.5. Remark. Given a group S-scheme (G, μ, e, i) and an S-scheme $s : T \to S$, we have an induced group T-scheme (G_s, μ_s, e_s, i_s), where $G_s = G \times_S T$ and μ_s, e_s, i_s are the morphisms induced by pullback of μ, e, i, respectively. Moreover, if $f : G \to H$ is a morphism of group S-schemes, pullback by s induces a morphism $f_s : G_s \to H_s$ of group T-schemes. Thus, every morphism $s : T \to S$ induces a functor $s^* : \mathbf{GrpSch}_S \to$

$\mathbf{GrpSch}_T$. On the other hand, every group T-scheme G can also be viewed as a group S-scheme, and so we see we also have a functor $s_* : \mathbf{GrpSch}_T \to \mathbf{GrpSch}_S$. In particular, we have a natural functor $\mathbf{GrpSch}_S \to \mathbf{GrpSch}_{\operatorname{Spec} \mathbf{Z}}$.

3. Groups and Group Schemes

We wish to naturally associate to any *group* a group S-scheme. We first define a functor $F : \mathbf{Set} \to \mathbf{Sch}_S$, as follows. On objects, we define $F(A) = \coprod_{a \in A} S_a$, where S_a is simply the scheme S labeled with the index a for component-tracking purposes. Given a set map $f : A \to B$, we define $F(f) : F(A) \to F(B)$ to be the product of the identity morphisms of S sending component a of $F(A) = \coprod_a S_a$ to component $f(a)$ of $F(B) = \coprod_b S_b$. Let $\tilde{F}$ denote the composition of F with the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$.

3.1. Proposition. — *There is a factorization of the functor $\tilde{F} : \mathbf{Grp} \to \mathbf{Sch}_S$ through the forgetful functor $\mathbf{GrpSch}_S \to \mathbf{Sch}_S$.*

Proof. Suppose $\mathcal{G}$ is a group, and write $G = \tilde{F}(\mathcal{G}) = \coprod_{g \in \mathcal{G}} S_g$. To define a group structure on G, first observe that for every $g_1, g_2 \in \mathcal{G}$ we have a natural inclusion $S_{g_1} \times_S S_{g_2} \cong S_{g_1 g_2} \hookrightarrow G$, and these inclusions collectively define an S-morphism $\mu_G : G \times_S G \to G$. The "identity" S-morphism is given by $e_G : S \cong S_1 \hookrightarrow G$, where 1 is the identity element of $\mathcal{G}$, and the "inverse" S-morphism is given by the collection of S-morphisms $\{i_g : S_g \to S_{g^{-1}} \hookrightarrow G\}_{g \in \mathcal{G}}$. The commutativity of the "associativity," "identity," and "inverse" diagrams follows immediately from the corresponding properties of $\mathcal{G}$.

Now suppose $f : \mathcal{G} \to \mathcal{H}$ is a group homomorphism, and write $H = \tilde{F}(\mathcal{H})$. Since $f(1_{\mathcal{G}}) = 1_{\mathcal{H}}$, we have a commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{\sim} & S_{1_{\mathcal{H}}} & \hookrightarrow & H \\ {\scriptstyle\sim}\downarrow & \nearrow & & \nearrow {\scriptstyle F(f)} & \\ S_{1_{\mathcal{G}}} & \hookrightarrow & G, & & \end{array}$$

where the composition of the left vertical and bottom horizontal arrows is e_G, and the composition of the top horizontal arrows is e_H. Similarly, since $f(g_1 g_2) = f(g_1) f(g_2)$ for every $g_1, g_2 \in \mathcal{G}$, we have commutative diagrams

$$\begin{array}{ccccc} S_{g_1} \times_S S_{g_2} & \longrightarrow & S_{f(g_1)} \times_S S_{f(g_2)} & \hookrightarrow & H \times_S H \\ {\scriptstyle\sim}\downarrow & & {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle \mu_H} \\ S_{g_1 g_2} & \longrightarrow & S_{f(g_1 g_2)} = S_{f(g_1) f(g_2)} & \hookrightarrow & H, \end{array}$$

which together assemble into the commutative diagram

$$\begin{array}{ccc} G \times_S G & \xrightarrow{F(f) \times F(f)} & H \times_S H \\ {\scriptstyle \mu_G}\downarrow & & \downarrow{\scriptstyle \mu_H} \\ G & \xrightarrow{F(f)} & H. \end{array}$$

Thus, $F(f)$ does indeed define a morphism of group S-schemes. $\square$

3.2. Remark. We have implicitly used the fact that $\coprod_{g_1,g_2\in\mathcal{G}}(S_{g_1}\times_S S_{g_2}) \cong (\coprod_{g_1\in\mathcal{G}} S_{g_1})\times_S (\coprod_{g_2\in\mathcal{G}} S_{g_2})$ in order to define the morphism $\mu : G\times_S G \to G$. It is not true in general that limits (e.g., products) and colimits (e.g., disjoint unions) commute, even in the case of *finite* limits. It is true, however, that *finite* limits and *filtered* colimits commute [Ma, IX.2,Thm.1]. When $\mathcal{G}$ is viewed as a category (namely, the category with a single object $\star$ and with $\mathrm{Hom}(\star,\star) = \mathcal{G}$) it is trivially seen to be filtered, and so we were correct in our implicit assumption.

3.3. Remark. The group S-scheme $\tilde{F}(\mathcal{G})$ is sometimes called the *constant group scheme* associated to $\mathcal{G}$, and is denoted $S_{\mathcal{G}}$. This name refers to the fact that it is the scheme representing the constant sheaf associated to $\mathcal{G}$.

4. Relatively Affine Group Schemes

Suppose $G \to S$ is an affine morphism, so that $G \cong \mathrm{Spec}_S\,\mathcal{A}$ for some sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$. Since the category of schemes affine over S is equivalent to the (opposite) category of sheaves of $\mathcal{O}_S$-algebras on S (via the global sections and relative spectrum functors), it should be straightforward to translate the data of a group structure on G into the corresponding data on $\mathcal{A}$. As we will see, the corresponding data is that of a co-$\mathcal{O}_S$-algebra structure on $\mathcal{A}$ compatible with the original $\mathcal{O}_S$-algebra structure.

We first consider the multiplication, identity, and inverse S-morphisms for G. As $G\times_S G \cong \mathrm{Spec}_S\,(\mathcal{A}\otimes_{\mathcal{O}_S}\mathcal{A})$, an S-morphism $\mu : G\times_S G \to G$ is equivalent to an $\mathcal{O}_S$-algebra morphism $\mu^\sharp : \mathcal{A} \to \mathcal{A}\otimes_{\mathcal{O}_S}\mathcal{A}$. Similarly, an S-morphism $e : S \to G$ is equivalent to an $\mathcal{O}_S$-algebra morphism $e^\sharp : \mathcal{A} \to \mathcal{O}_S$, and an S-morphism $i : G \to G$ is equivalent to an $\mathcal{O}_S$-algebra morphism $i^\sharp : \mathcal{A} \to \mathcal{A}$. It is reasonable to call $\mu^\sharp$, $e^\sharp$, and $i^\sharp$ the *co-multiplication*, *co-identity*, and *co-inverse* in $\mathcal{A}$, respectively.

In a similar manner, the three commutative diagrams required for (μ, e, i) to define a group structure on G are equivalent to the following three commutative diagrams of $\mathcal{O}_S$-algebras:

(i) (Co-associativity)

$$\begin{array}{ccc}
\mathcal{A}\otimes_{\mathcal{O}_S}\mathcal{A}\otimes_{\mathcal{O}_S}\mathcal{A} & \xleftarrow{1\otimes\mu^\sharp} & \mathcal{A}\otimes_{\mathcal{O}_S}\mathcal{A} \\
{\scriptstyle \mu^\sharp\otimes 1}\uparrow & & \uparrow{\scriptstyle \mu^\sharp} \\
\mathcal{A}\otimes_{\mathcal{O}_S}\mathcal{A} & \xleftarrow{\mu^\sharp} & \mathcal{A}.
\end{array}$$

(ii) (Co-identity)

$$\begin{array}{ccccccccc} \mathcal{A} & \xleftarrow{\sim} & \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{A} & \xleftarrow{e^\sharp \otimes 1} & \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} & \xrightarrow{1 \otimes e^\sharp} & \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_S & \xrightarrow{\sim} & \mathcal{A} \\ & {\scriptstyle 1} \nwarrow & & & \uparrow {\scriptstyle \mu^\sharp} & & & \nearrow {\scriptstyle 1} & \\ & & & & \mathcal{A}. & & & & \end{array}$$

(iii) (Co-inverse, or Antipode)

$$\begin{array}{ccccccccc} \mathcal{A} & \xleftarrow{\nabla} & \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} & \xleftarrow{i^\sharp \otimes 1} & \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} & \xrightarrow{1 \otimes i^\sharp} & \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} & \xrightarrow{\nabla} & \mathcal{A} \\ \uparrow & & & & \uparrow {\scriptstyle \mu^\sharp} & & & & \uparrow \\ \mathcal{O}_S & & \xleftarrow{\quad e^\sharp \quad} & & \mathcal{A} & & \xrightarrow{\quad e^\sharp \quad} & & \mathcal{O}_S \end{array}$$

An $\mathcal{O}_S$-algebra equipped with $\mathcal{O}_S$-algebra morphisms $(\mu^\sharp, e^\sharp)$ satisfying the first two commutative diagrams is usually called an *$\mathcal{O}_S$-bialgebra*. An $\mathcal{O}_S$-bialgebra possessing an $\mathcal{O}_S$-algebra morphism $i^\sharp$ satisfying the third commutative diagram is usually called a *Hopf $\mathcal{O}_S$-algebra*.

4.1. Remark. If $\mathcal{G}$ is a group, then $S_{\mathcal{G}}$ is affine over S if and only if $\mathcal{G}$ is finite.

5. Subgroup Schemes and Quotients

It seems likely one might be interested in studying subgroups of group schemes, so we make the following definition.

5.1. Definition. A *subgroup S-scheme* of a group S-scheme (G, μ_G, e_G, i_G) is a group S-scheme (H, μ_H, e_H, i_H) where H is a subscheme of G and the two actions are compatible; i.e., the following diagrams commute:

$$\begin{array}{ccc} H \times_S H & \xrightarrow{\mu_H} & H \\ \downarrow & & \downarrow \\ G \times_S G & \xrightarrow{\mu_G} & G \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{e_H} & H \\ & {\scriptstyle e_G} \searrow & \downarrow \\ & & G \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{i_H} & H \\ \downarrow & & \downarrow \\ G & \xrightarrow{i_G} & G. \end{array}$$

For notational simplicity, we will often simply write $H \leq G$ to indicate that (H, μ_H, e_H, i_H) is a subgroup scheme of (G, μ_G, e_G, i_G), and we will say H is a *subgroup* of G.

5.2. Remark. By Yoneda's lemma, the data of a subgroup S-scheme structure on a subscheme H of a group scheme G corresponds to the data of natural subgroup structures of the subsets $H(T)$ of the groups $G(T)$ for all S-schemes T.

As in the usual case with subgroups, it is not always the case that one can define a quotient group scheme G/H. It seems that there has been a good deal of work on determining precisely when a subgroup H of a group scheme G induces a quotient group scheme G/H. The following is a sampling of what is known:

(i) The functor $T \mapsto G(T)/H(T)$ might not be a sheaf, and its sheafification might not be representable (by a scheme);
(ii) If H is finite, flat, and closed in G, then the associated sheaf in (i) is representable by a scheme, and that scheme admits a canonical left G-action (by translation). If the restriction of this action to H is trivial, then H is called *normal* and the quotient G/H admits a natural group structure;
(iii) If H is closed in G and both are affine, then the quotient is representable.

For a survey of these types of results, see [BB2].

6. ACTIONS ON SCHEMES

6.1. Definition (cf. [Mu, Def. 0.3]). Suppose G is a group S-scheme and X is an S-scheme. A *left action of* G *on* X is an S-morphism $\sigma : G \times_S X \to X$ making the following diagrams commute:

(i) (Compatibility with group structure)

$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{1\times\sigma} & G \times_S X \\ \downarrow{\scriptstyle \mu\times 1} & & \downarrow{\scriptstyle \sigma} \\ G \times_S X & \xrightarrow{\sigma} & X. \end{array}$$

(ii) (Identity action)

$$\begin{array}{ccccc} X & \xrightarrow{\sim} & S \times_S X & \xrightarrow{e\times 1} & G \times_S X \\ & \searrow^{1} & & & \downarrow{\scriptstyle \sigma} \\ & & & & X. \end{array}$$

We similarly define a *right action* of G on X in the obvious way.

6.2. Example. A group S-scheme G naturally acts on itself (both on the left and the right) via the multiplication morphism $\mu : G \times_S G \to G$.

6.3. Remark. By Yoneda's lemma, an action of G on X is equivalent to a natural action of the group $G(T)$ on the set $X(T)$ for every S-scheme T.

For actions of *groups* on S-schemes, it is natural to make the following definition.

6.4. Definition. Suppose $\mathcal{G}$ is a group and X is an S-scheme. A *left action of* $\mathcal{G}$ *on* X is a group homomorphism $\sigma : \mathcal{G} \to \mathrm{Aut}_S(X)$, where the group operation in $\mathrm{Aut}_S(X)$ is given by composition, i.e., $f_1 \cdot f_2 = f_1 \circ f_2$. The action is *faithful* if σ is injective.

We similarly define a *right action* of $\mathcal{G}$ on X as a group homomorphism $\sigma' : \mathcal{G} \to \mathrm{Aut}_S(X)^{\mathrm{op}}$, where the group operation in $\mathrm{Aut}_S(X)^{\mathrm{op}}$ is reverse composition, i.e., $f_1 \star f_2 = f_2 \circ f_1$.

Suppose X is an S-scheme and $\mathcal{G}$ is a group. By the following proposition, we may interchangeably consider actions of $\mathcal{G}$ on X and actions of $S_{\mathcal{G}}$ on X.

6.5. PROPOSITION. — *Suppose X is an S-scheme and $\mathcal{G}$ is a group. Then the data of a left (resp. right) action of $\mathcal{G}$ on X is equivalent to the data of a left (resp. right) action of $S_{\mathcal{G}}$ on X.*

Proof. For convenience, let us write $G = S_{\mathcal{G}}$. Observe that we have natural isomorphisms

$$\mathrm{Mor}_S(G \times_S X, X) \cong \mathrm{Mor}_S(\coprod_{g \in \mathcal{G}} X_g, X) \cong \prod_{g \in \mathcal{G}} \mathrm{Mor}_S(X_g, X) \cong \prod_{g \in \mathcal{G}} \mathrm{End}_S(X).$$

We therefore have a natural bijection between the collection of S-morphisms $\sigma : G \times_S X \to X$ and the collection of set maps $\psi : \mathcal{G} \to \mathrm{End}_S(X)$. We claim that under this bijection the morphisms σ that define left actions of G on X correspond precisely to those set maps ψ that define left actions of $\mathcal{G}$ on X. First note that we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\sim} & S \times_S X & \xrightarrow{e \times 1} & G \times_S X \cong \coprod_{g \in \mathcal{G}} X_g \\ & \searrow^{\psi(e)} & & & \downarrow \sigma \; \prod_g \psi(g) \\ & & & & X, \end{array}$$

and so condition (ii) for σ to define a left action is equivalent to the requirement that $\psi(e) = 1$. Next, observe that commutativity of the diagram

$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{1 \times \sigma} & G \times_S X \\ \downarrow \mu \times 1 & & \downarrow \sigma \\ G \times_S X & \xrightarrow[\sigma]{} & X \end{array}$$

is equivalent to commutativity of the diagram

$$\begin{array}{ccc} \coprod_{g_1, g_2 \in \mathcal{G}} X_{g_1, g_2} & \xrightarrow{\coprod_{g_1} \prod_{g_2} \psi(g_2)} & \coprod_{g_1 \in \mathcal{G}} X_{g_1} \\ \downarrow & & \downarrow \prod_{g_1} \psi(g_1) \\ \coprod_{g \in \mathcal{G}} X_g & \xrightarrow[\prod_g \psi(g)]{} & X, \end{array}$$

where the left vertical arrow is given by $\prod_{g_1, g_2}(X_{g_1, g_2} \xrightarrow{\sim} X_{g_1 g_2} \hookrightarrow \coprod_g X_g)$. In this latter diagram, the composition of the top horizontal and right vertical maps is the morphism $\prod_{g_1, g_2}(\psi(g_1) \circ \psi(g_2)) : \coprod_{g_1, g_2} X_{g_1, g_2} \to X$, while the composition of the left vertical and bottom horizontal maps is the morphism $\prod_{g_1, g_2} \psi(g_1 g_2) : \coprod_{g_1, g_2} X_{g_1, g_2} \to X$. Thus, commutativity of this diagram is equivalent to the statement that $\psi(g_1) \circ \psi(g_2) = \psi(g_1 g_2)$ for every $g_1, g_2 \in \mathcal{G}$.

So, we've shown that the morphisms σ that define left actions of G on X correspond precisely to the maps $\psi : \mathcal{G} \to \mathrm{End}_S(X)$ that are morphisms of monoids. Since $\mathcal{G}$ is a group, all such morphisms factor uniquely through the group of invertible elements of $\mathrm{End}_S(X)$, namely $\mathrm{Aut}_S(X)$. Thus, the morphisms σ that define left actions of G on X correspond precisely to the group homomorphisms $\psi : \mathcal{G} \to \mathrm{Aut}_S(X)$, i.e., the morphisms ψ that define left actions of $\mathcal{G}$ on X. □

7. Induced Action of Points

Suppose (G, μ, e, i) is a group S-scheme. For each $g \in G(S)$, define $\sigma(g)$ as the composition

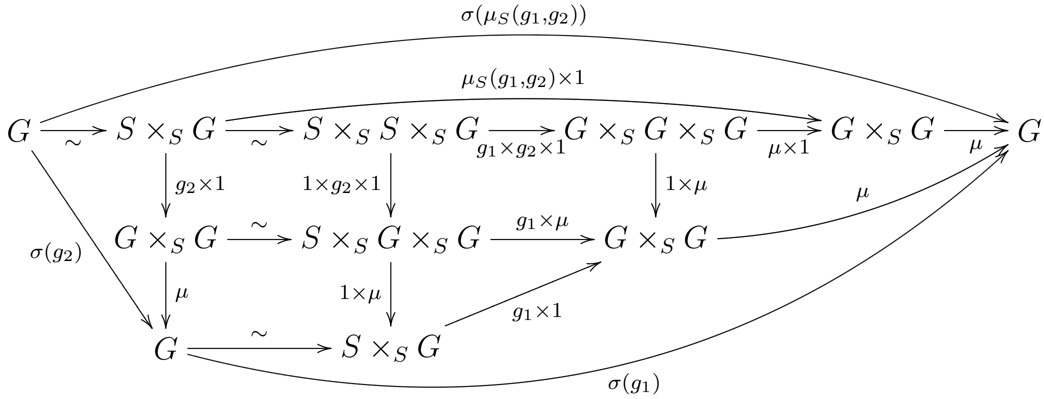
$$G \xrightarrow{\sim} S \times_S G \xrightarrow{g \times 1} G \times_S G \xrightarrow{\mu} G.$$

This defines a set map $\sigma : G(S) \to \mathrm{End}_S(G)$.

7.1. Proposition. — *The map σ defines a faithful left action of $G(S)$ on G.*

Proof. Suppose for the moment that σ does indeed define a group homomorphism to $\mathrm{Aut}_S(G)$. Then the kernel of σ consists precisely of those morphisms $g : S \to G$ satisfying the "left identity" diagram for G, and so by Remark 2.4, it follows that σ must be injective.

Now suppose $g_1, g_2 \in G(S)$. The equality $\sigma(\mu_S(g_1, g_2)) = \sigma(g_1) \circ \sigma(g_2)$ follows from the commutative diagram below (which is less complicated than it might at first appear):



As $\sigma(e) = 1$ (by the "(left) identity" diagram), we've thus established that $\sigma : G(S) \to \mathrm{End}_S(G)$ is a morphism of monoids. As $G(S)$ is a group, the morphism σ automatically factors through the group of invertible elements in $\mathrm{End}_S(G)$, which is precisely $\mathrm{Aut}_S(G)$. $\square$

7.2. Remark. For each $g \in G(S)$, we can similarly define $\sigma'(g)$ as the composition

$$G \xrightarrow{\sim} G \times_S S \xrightarrow{1 \times g} G \times_S G \xrightarrow{\mu} G.$$

The above argument, mutatis mutandis, proves that this defines a faithful right action of $G(S)$ on G.

8. Extending Actions from Subgroup Schemes

Suppose a subgroup H of a group scheme G acts on a scheme X. It is natural to ask how one might extend the action of H to an action of G. There are two obvious approaches. The first is to attempt to construct a "minimal" extension of X on which G naturally acts, akin to the so-called "mixing construction" of Borel. By this we mean the following: we let G act on left of $G \times_S X$ via the natural left multiplication on the first factor, and let H act on the left via the inverse of right multiplication on the first factor and the given action on the second factor. In the case of spaces, this is written as $h \cdot (g,x) = (gh^{-1}, h \cdot x)$. Noting that the G and H actions commute, one then quotients $G \times_S X$ by the action of H, and calls the resulting quotient G-scheme $G \times_H X$. It is not clear in this construction, however, that the necessary quotient scheme actually exists.

A second obvious approach is to interpret the problem functorially, and to ask whether one can find a universal H-morphism $X \to X'$ to a G-scheme X' (where the H-action on X' is the one restricting from the given G-action). Again, the existence of the desired universal object is not clear.

In both of these approaches, the answer is not completely satisfactory, at least if one restricts oneself to the category of schemes. If one allows oneself to work with algebraic spaces or stacks, satisfactory results can be stated. For example, we have the following:

8.1. THEOREM ([BB, Cor. 2]. — *Let X be an algebraic space with an action of an algebraic group H and let G be an affine algebraic group containing H as a closed subgroup. Then there exists a good geometric quotient*

$$G \times X \to (G \times X)/H = G \times_H X,$$

where $G \times_H X$ is an algebraic G-space with an action of G induced by left translations on G. If X can be covered by H-invariant quasiprojective open subsets, then $G \times_H X$ is an algebraic variety.

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