A NOTE ON POINTS OF ALGEBRAIC STACKS

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For ease of reference, we give here proofs of several basic results on Artin stacks. First, we show that every quasicompact morphism between Artin stacks satisfies a closure property sometimes known as "adherence." We then use that property to show that every irreducible Artin stack has a unique generic point, from which we in turn deduce that every quasicompact Artin stack contains a closed point. These statement are not new (see [LMB]), but their proofs are short and illustrate a guiding principle: the general language of stacks often makes it straightforward to generalize results about affine schemes directly to Artin stacks. We begin with a simple lemma.

0.1. LEMMA. — *Every nonzero commutative ring contains both a minimal prime ideal and a maximal proper ideal.*

Proof. Let S be the set of proper ideals of the nonzero commutative ring R. Then S is nonempty (as it contains the zero ideal) and partially-ordered (by inclusion). For any chain $\{I_\alpha\}_{\alpha\in\Lambda}$ in S, let $I = \sum_{\alpha\in\Lambda} I_\alpha$. Then I is a I is proper ideal; for if it were not proper, then we would have $1 \in \sum_{i=1}^{n} I_{\alpha_i}$ for some $\alpha_1, \ldots, \alpha_n$, and hence $1 \in \max\{I_{\alpha_1}, \ldots, I_{\alpha_n}\} \subsetneq R$. So, every chain in S has an upper bound, and hence Zorn's lemma implies that S has a maximal element.

To prove the existence of a minimal prime ideal, let T be the set of prime ideals of R. Then T is nonempty (as it contains the maximal ideal) and partially-ordered (by reverse-inclusion). For any chain $\{\mathfrak{p}_\alpha\}_{\alpha\in\Lambda}$, let $\mathfrak{p} = \cap_{\alpha\in\Lambda}\mathfrak{p}_\alpha$. We claim $\mathfrak{p}$ is a prime ideal. Indeed, suppose $x, y \in R$ are such that $xy \in \mathfrak{p}$ and $x \notin \mathfrak{p}$. Then there exists some α_0 such that $x \notin \mathfrak{p}_{\alpha_0}$, and hence $x \notin \mathfrak{p}_\alpha$ for every $\alpha \geq \alpha_0$ (by which we mean $\mathfrak{p}_\alpha \supseteq \mathfrak{p}_{\alpha_0}$). Since $xy \in \mathfrak{p}_\alpha$ for every α–in particular for every $\alpha \geq \alpha_0$–we must therefore have $y \in \mathfrak{p}_\alpha$ for every $\alpha \geq \alpha_0$. This implies $y \in \cap_{\alpha\geq\alpha_0}\mathfrak{p}_\alpha = \mathfrak{p}$. So, $\mathfrak{p}$ is prime, and hence is an upper bound for the given chain. Zorn's lemma then gives the desired minimal prime. □

0.2. COROLLARY. — *Every affine scheme has both a closed point and a maximal point.*

0.3. LEMMA. — *Suppose $f : X \to Y$ is a morphism of affine schemes. Then $y \in \overline{f(X)}$ if and only if $y \in \overline{f(x)}$ for some $x \in X$.*

Proof. One direction is immediate, since $\overline{f(x)} \subseteq \overline{f(X)}$ for any point $x \in X$. For the converse, we may assume f is dominant (replacing Y by the affine subscheme $\overline{f(X)}$ if necessary). If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, then f corresponds to an injective ring homomorphism $\phi : B \hookrightarrow A$. If $y \in Y$ corresponds to the prime ideal $\mathfrak{q} \subset B$, then the set S of

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points $x \in X$ such that $y \in \overline{f(x)}$ corresponds bijectively to the set of prime ideals of the ring $A \otimes_B B_{\mathfrak{q}}$. Since $B \hookrightarrow A$, we have $B_{\mathfrak{q}} \hookrightarrow A \otimes_B B_{\mathfrak{q}}$; in particular, the ring $A \otimes_B B_{\mathfrak{q}}$ is nonzero. By the previous lemma, it contains a maximal (hence prime) ideal, and hence S is nonempty. □

We now directly generalize these results to Artin stacks. We begin with a property sometimes called "adherence."

0.4. PROPOSITION [LMB, Prop. 5.7]. — *Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a quasicompact morphism of Artin stacks. Then $y \in \overline{f(|\mathcal{X}|)}$ if and only if $y \in \overline{f(x)}$ for some point $x \in |\mathcal{X}|$.*

Proof. As in the case of affine schemes, one direction is immediate. For the converse, since the statement is local on $\mathcal{Y}$ we may suppose $\mathcal{Y}$ (and hence also $\mathcal{X}$) is quasi-compact. Let $q : Y \to \mathcal{Y}$ be an atlas for $\mathcal{Y}$, and let $f' : \mathcal{X}' \to Y$ be the corresponding base change of f. Since q is open, we have $\overline{q^{-1}(Z)} = q^{-1}(\overline{Z})$ for every subset Z of $|\mathcal{Y}|$. Combined with the fact that $q^{-1}(f(|\mathcal{X}|)) = f'(|\mathcal{X}'|)$, we may (upon replacing $\mathcal{Y}$ with Y and f with f') assume $\mathcal{Y} = Y$ is a scheme. Finally, if $p : X \to \mathcal{X}$ is an atlas for $\mathcal{X}$ with a quasi-compact scheme X, then $f(|\mathcal{X}|) = (f \circ p)(|X|)$, and we may therefore replace $\mathcal{X}$ with X. We've thus reduced to the case of quasi-compact schemes, which in turn reduces immediately to the case of affine schemes, from which the statement follows by the previous lemma. □

0.5. PROPOSITION [LMB, Cor. 5.7.2]. — *If $\mathcal{X}$ is an Artin stack, then every irreducible closed subset of $|\mathcal{X}|$ has a unique generic point.*

Proof. It suffices to prove $|\mathcal{X}|$ has a unique generic point in the case $\mathcal{X}$ is irreducible. Let $p : X \to \mathcal{X}$ be a smooth presentation by a scheme X. Since smooth morphisms are open, the image of any nonempty open affine subscheme $U \subseteq X$ is a nonempty open (hence irreducible) substack of $\mathcal{U} \subseteq \mathcal{X}$. Now, the irreducible topological space $|\mathcal{X}|$ has a unique generic point if and only if there is a nonempty open (irreducible) subset $|\mathcal{U}| \subseteq |\mathcal{X}|$ which has a unique generic point. Replacing X by U and $\mathcal{X}$ by $\mathcal{U}$, we may thus assume X is affine and p is an open surjection. We can then take x to be any maximal point of the affine scheme X, and let $\eta = p(x) \in |\mathcal{X}|$.

We claim η is a maximal point of $|\mathcal{X}|$. First note that $|\mathcal{X}|$ has a base of quasi-compact opens (since $|X|$ has such a base and the morphism p is open), so we may assume $\mathcal{X}$ is quasi-compact. Suppose $\eta' \in |\mathcal{X}|$ is a generization of η. Since p is open, we have $\overline{p^{-1}(Z)} = p^{-1}(\overline{Z})$ for every subset Z of $|\mathcal{X}|$; in particular, taking $Z = \{\eta'\}$, we see that x is in the closure of $p^{-1}(\eta')$. But $p^{-1}(\eta)$ is the image of the quasi-compact morphism $u : X' \to X$, where $X' = \operatorname{Spec} K \times_{i,\mathcal{X},p} X$ and where $i : \operatorname{Spec} K \to \mathcal{X}$ is a representative of the point η'. By the previous proposition, it follows that x is in the closure of a point $x' \in p^{-1}(\eta')$, and hence by maximality that $x = x'$. Therefore $\eta = p(x) = p(x') = \eta'$.

We now show that, for every nonempty open substack $\mathcal{U}$ of $\mathcal{X}$, we have $\eta \in |\mathcal{U}|$. Since $|\mathcal{X}|$ has a base of quasi-compact opens, we may assume $\mathcal{U}$ is quasi-compact. As $\mathcal{X}$ is quasi-separated, the inclusion of $\mathcal{U}$ in $\mathcal{X}$ is quasi-compact, and since $\mathcal{X}$ is irreducible, the image is dense. It then follows from the previous proposition that η is in the closure of a point $u \in |\mathcal{U}|$, and by the maximality of η this implies $\eta = u \in |\mathcal{U}|$. We've thus proven that η is a generic point of $\mathcal{X}$; uniqueness follows immediately from the maximality of η. □

0.6. PROPOSITION. — *Every quasicompact Artin stack contains a closed point.*

Proof. Let $\mathcal{X}$ be a quasicompact Artin stack, and let $p : \operatorname{Spec} A \to \mathcal{X}$ be a smooth surjection from an affine scheme. We claim first that $\mathcal{X}$ contains a minimal nonempty closed substack. Indeed, let S be the set of all nonempty closed substacks of $\mathcal{X}$. Then S is nonempty, as it contains $\mathcal{X}$, and is partially-ordered by reverse-inclusion. Let $\{\mathcal{Z}_\alpha\}_{\alpha\in\Lambda}$ be any chain in S, and let $\mathcal{Z} = \cap_{\alpha\in\Lambda}\mathcal{Z}_\alpha$. Then $\mathcal{Z}$ is a closed substack of $\mathcal{X}$, and we claim $\mathcal{Z}$ is nonempty. Indeed, since p is surjective, we have that $\{p^{-1}(\mathcal{Z}_\alpha)\}_{\alpha\in\Lambda}$ is a chain of nonempty closed subschemes of the affine scheme $\operatorname{Spec} A$, and hence corresponds to a chain $\{I_\alpha\}_{\alpha\in\Lambda}$ of proper ideals of A. By the proof of the lemma, the ideal $\sum_{\alpha\in\Lambda} I_\alpha$ is proper, and hence the subscheme $\cap_{\alpha\in\Lambda} p^{-1}(\mathcal{Z}_\alpha)$ is nonempty. It follows that $|\mathcal{Z}| = p(\cap_{\alpha\in\Lambda} p^{-1}(|\mathcal{Z}_\alpha|))$ is nonempty.

By Zorn's lemma, it therefore follows that S has a minimal element, i.e., there exists a minimal nonempty closed substack $\mathcal{Z}_0 \subseteq \mathcal{X}$. By minimality, $\mathcal{Z}_0$ must be irreducible, and hence (by the previous proposition) $\mathcal{Z}_0$ has a unique generic point, $\xi \in |\mathcal{Z}_0|$. If $x \in |\mathcal{Z}_0|$ is any point, then $\overline{\{x\}} \subseteq \mathcal{Z}_0$ is a nonempty closed substack, and so by minimality we must have $\overline{\{x\}} = \mathcal{Z}_0$. By the uniqueness of ξ, we then have $x = \xi$. We've thus shown that $|\mathcal{Z}_0| = \{\xi\}$, and hence ξ is a closed point of $\mathcal{X}$. □

REFERENCES

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