S_3-COVERS OF SCHEMES

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Abstract

The theory of branched group covers of algebraic schemes has been well studied in abelian case, and has proven useful for constructing schemes with interesting geometric properties. Here we extend the theory to the group S_3, ultimately obtaining a local characterization of flat S_3-covers.

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Chapter 1

Introduction

In the study of algebraic geometry, it is useful — and often essential — to have at one's disposal a vast repertoire of examples, both to understand the structure of general schemes as well as to understand the pathologies which can occur. The theory of branched group covers is particularly well-suited to producing these examples.

To quickly summarize the basic facts of the theory, suppose G is a finite group and k is a field of characteristic not dividing the order of G (and for the purpose of this introduction, also assume k is algebraically closed). A G-cover of a scheme X/k is a scheme Y/k together with a faithful G-action on Y and a finite morphism $\pi : Y \to X$ which identifies X with the geometric quotient Y/G. Given a flat G-cover, there exists a decomposition $\pi_*\mathcal{O}_Y = \oplus_\rho \mathcal{F}_\rho$, indexed by the irreducible k-representations of G, where the $\mathcal{F}_\rho$ are locally free $\mathcal{O}_X[G]$-modules with G-actions closely related to the indexing representations ρ. Conversely, to construct a cover given an appropriate collection of locally free $\mathcal{O}_X[G]$-modules $\{\mathcal{F}_\rho\}_\rho$, one defines an algebra structure on $\mathcal{A} = \oplus_\rho \mathcal{F}_\rho$ compatible with the given G-actions. One then obtains the G-cover $\pi : \mathbf{Spec}_X \mathcal{A} \to X$.

In the case of abelian groups, the foundations of the theory were laid by Pardini in [12]. In this case, the $\mathcal{O}_X[G]$-submodule $\mathcal{F}_\chi$ is the invertible χ-eigensheaf of $\pi_*\mathcal{O}_Y$, defined by the collection of sections on which the group acts as multiplication by the character χ. The algebra structure on $\pi_*\mathcal{O}_Y$ is determined by a compatible collection of morphisms $\mathcal{F}_\chi \otimes \mathcal{F}_{\chi'} \to \mathcal{F}_{\chi\chi'}$, which is equivalent to a collection of global sections of the sheaves $\mathcal{F}_\chi^{-1} \otimes \mathcal{F}_{\chi'}^{-1} \otimes \mathcal{F}_{\chi\chi'}$. As it turns out, these sections are closely related to the branch divisor of the cover. In fact, given the invertible sheaves $\{\mathcal{F}_\chi\}_\chi$, to construct a G-cover one may replace the explicit definition of the algebra structure instead with a specification of branching data. As long as a simple "covering condition" is satisfied, one can then construct a G-cover.

This reduction of the construction of a cover to the specification of the branching data, coupled with an explicit understanding of the invariants and deformations of the cover, allows one to construct schemes Y encoding geometry of divisors in the base schemes X. For example, if X is a surface, one can use geometrically interesting configurations of curves in X to construct new surfaces whose intrinsic geometry reflects the geometry of the configuration.

This technique has proven remarkably fruitful. For example, a standard result in the theory of complex surfaces is the so-called Bogomolov-Miyaoka-Yau inequality. It states that if X is a smooth, complex surface of general type, then $\frac{K_X^2}{\chi(\mathcal{O}_X)} \leq 9$ ([1],[10],[14]). This inequality is sharp, and Hirzebruch has produced examples of equality by constructing abelian covers of $\mathbb{P}^2$ branched over "extreme" configurations of lines [6]. The inequality is known to fail in positive characteristic [8], and infinite families of counterexamples can be produced using abelian covers branched over configurations of lines occuring only in positive characteristic [2].

The usefulness of the abelian theory motivates the extension of the theory to nonabelian groups. The situation is then significantly more complicated, likely even intractable in general, but the group S_3 at least is within reach. Indeed, the aim of this thesis is a complete understanding of the local nature of S_3-covers, the end result of which will likely serve as the groundwork for future investigations.

This thesis is organized as follows. Chapter 2 serves mainly to establish terminology, but also includes some preliminary results about general G-actions used in later chapters. Chapter 3 is an overview of the basic theory of G-covers, meant not as an exhaustive study but rather to establish the results necessary for the subsequent analysis of S_3-covers. Chapter 4 is the technical heart of the analysis, the focus of which is obtaining a local characterization of flat S_3-covers. This characterization is obtained via two lengthy calculations. After obtaining a local understanding of the covers, we briefly examine the ramification locus, as well as the relationship between S_3-covers and triple covers. We conclude in Chapter 5 with a brief outline of areas of future research now amenable to study, given our new understanding of the local structure of S_3-covers.

Chapter 2

Preliminaries

Summary: We recall the definition of a G-action on a scheme, for a finite group G. We also examine a standard decomposition of $\mathcal{O}_X[G]$-modules on a scheme X over a field k of characteristic not dividing the order of G.

2.1 Group actions

Definition 2.1.1. Let G be a finite group. Given a category C and object X of C, a G**-action on** X is a group homomorphism

$$\mu : G \to \mathrm{Aut}_C(X).$$

The action is **faithful** if μ is injective.

We will generally deal with the category of schemes over a given field k, or sheaves on such a scheme. We will refer to a sheaf of $\mathcal{O}_X$-modules with a G-action as an $\mathcal{O}_X[G]$-module.

Remark 2.1.2. More generally, if $\mathcal{G}$ is a group scheme over S, with group operation given by $m : \mathcal{G} \times_S \mathcal{G} \to \mathcal{G}$, then a $\mathcal{G}$-action on an S-scheme X is a morphism

$$\mu : \mathcal{G} \times_S X \to X$$

such that the diagram

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{G} \times X & \xrightarrow{m \times 1} & \mathcal{G} \times X \\ {\scriptstyle 1\times\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{G} \times X & \xrightarrow[\mu]{} & X \end{array}$$

commutes.

Given a finite group G, the associated group scheme is $\mathcal{G} = \coprod_{g \in G} S$, and so a morphism $\mu : \mathcal{G} \times X \to X$ corresponds to a collection of morphisms $\{\mu(g) : X \to X\}_{g \in G}$. The commutativity of the above diagram is then equivalent to the statement $\mu(gh) = \mu(g) \circ \mu(h)$ for every $g, h \in G$; i.e., μ defines a group homomorphism $\mu : G \to \mathrm{Aut}_S(X)$. Thus, in the case of finite groups, the general definition agrees with the definition given above.

Example 2.1.3. Suppose X, Y are schemes and ν a G-action on Y. Let $\pi : Y \to X$ be a G-equivariant morphism of schemes. We will show there is an induced G-action on the sheaf $\pi_* \mathcal{O}_Y$ of $\mathcal{O}_X$-algebras.

First note that for every $g \in G$, we have a commutative diagram of schemes

$$\begin{array}{ccccc} Y & & \xrightarrow[\simeq]{\nu(g)} & & Y \\ & {\scriptstyle \pi}\searrow & & \swarrow{\scriptstyle \pi} & \\ & & X & & \end{array},$$

and hence a commutative diagram of sheaves of $\mathcal{O}_X$-algebras on X

$$\begin{array}{ccccc} \pi_* \mathcal{O}_Y = \pi_* \nu(g)_* \mathcal{O}_Y & & \xleftarrow[\simeq]{\pi_*(\nu(g)^\sharp)} & & \pi_* \mathcal{O}_Y \\ & {\scriptstyle (\pi \circ \nu(g))^\sharp = \pi^\sharp}\nwarrow & & \nearrow{\scriptstyle \pi^\sharp} & \\ & & \mathcal{O}_X & & \end{array}.$$

This defines a map of sets

$$\tilde{\nu} : G \longrightarrow \mathrm{Aut}_{\mathcal{O}_X}(\pi_* \mathcal{O}_Y) .$$

$$g \longmapsto \pi_*(\nu(g^{-1})^\sharp)$$

The presence of the inverse is required for $\tilde{\nu}$ to define a group homomorphism. Indeed, since ν is a group homomorphism, given $g, h \in G$, we have

$$\nu((gh)^{-1}) = \nu(h^{-1}g^{-1}) = \nu(h^{-1}) \circ \nu(g^{-1}).$$

Hence, the associated sheaf morphisms satisfy

$$\nu((gh)^{-1})^{\sharp} = (\nu(h^{-1})_* \nu(g^{-1})^{\sharp}) \circ \nu(h^{-1})^{\sharp},$$

and so

$$\begin{aligned}\tilde{\nu}(gh) &= \pi_*(\nu(h^{-1})_* \nu(g^{-1})^{\sharp}) \circ \pi_* \nu(h^{-1})^{\sharp} \\ &= \pi_* \nu(g^{-1})^{\sharp} \circ \pi_* \nu(h^{-1})^{\sharp} \\ &= \tilde{\nu}(g) \circ \tilde{\nu}(h).\end{aligned}$$

Remark 2.1.4. When there is no risk of confusion, we will suppress the notation for ν, writing simply g for the automorphism $\nu(g)$.

Example 2.1.5. Suppose X is a scheme, $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-algebras, and $\tilde{\nu}$ is a G-action on $\mathcal{F}$. We will show there is an induced G-action on the scheme $Y := \mathbf{Spec}_X \mathcal{F}$, and that the induced G-action on $\pi_* \mathcal{O}_Y \cong \mathcal{F}$ agrees with the original G-action on $\mathcal{F}$.

Fix any $g \in G$. Then $\tilde{\nu}(g^{-1})$ is an automorphism of $\mathcal{F}$ as a sheaf of $\mathcal{O}_X$-algebras. In particular, given any open $U \subset X$, we have an automorphism $\tilde{\nu}(g^{-1})(U) : \mathcal{F}(U) \to \mathcal{F}(U)$ of $\mathcal{O}_X(U)$-algebras, and these morphisms agree on intersections. This induces an automorphism of $\operatorname{Spec} \mathcal{F}(U)$ over U, and these glue to give an automorphism $\nu(g) : Y \to Y$ over X. This construction is clearly inverse to the construction in the previous example.

Caution. Note again that, because of the contravariance of Spec, it was necessary to induce the scheme automorphism corresponding to g from the sheaf automorphism corresponding to g^{-1} (to ensure ν defined a homomorphism).

2.2 Standard decomposition of $\mathcal{O}_X[G]$-modules

Suppose $k[G]$ has a decomposition into left ideals

$$k[G] = \bigoplus_{i=1}^{n} B_i$$

such that if $e = \sum_i e_i$ with $e_i \in B_i$, then each e_i lies in the center of $k[G]$, $e_i^2 = e_i$, and $e_i e_j = 0$ for $i \neq j$.

Remark 2.2.1. If k is algebraically closed, then by Maschke's theorem such a decomposition always exists. In this case, the B_i are the so-called simple components of $k[G]$, and correspond to irreducible k-representations ρ_i of G. Moreover, the element e_i is given explicitly by

$$e_i = \frac{\deg \chi_i}{|G|} \sum_{g \in G} \chi_i(g^{-1})g,$$

where χ_i is the character afforded by the representation ρ_i ([13, §8.3, 8.5]). As we'll see in Chapter 4, in the case of the group S_3 the above formula holds even when k is not algebraically closed.

Now, suppose A is a k-algebra and M is any left $A[G]$-module. Since e_i lies in the center of $k[G]$, multiplication by e_i defines a left $A[G]$-module morphism $\mu_{e_{ij}} : M \to M$, which is a projection since $e_i^2 = e_i$. Let M_i denote the image submodule. Since $e = \sum_i e_i$ acts as the identity on M, we have $M = \sum_i M_i$, and since $e_i e_j = 0$ for $i \neq j$, we actually have $M = \oplus_i M_i$. Let us call this the **standard decomposition** of M.

The following proposition ensures we can sheafify these notions.

Proposition 2.2.2. *Let X be any k-scheme and $\mathcal{F}$ any quasicoherent $\mathcal{O}_X[G]$-module. Then there are $\mathcal{O}_X[G]$-submodules $\mathcal{F}_i \subset \mathcal{F}$ together with a direct sum decomposition $\mathcal{F} = \oplus_i \mathcal{F}_i$ such that for every open $U \subset X$, $\mathcal{F}(U) = \oplus_i \mathcal{F}_i(U)$ is the standard decomposition of $\mathcal{F}(U)$.*

Proof. We can k-linearly extend the G-action

$$\tilde{\mu} : G \to \mathrm{Aut}_{\mathcal{O}_X}(\mathcal{F})$$

to a ring homomorphism

$$\tilde{\mu} : k[G] \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{F}).$$

Define $\phi_i := \tilde{\mu}(e_i)$. For every open set $U \subset X$, the morphism $\phi_i(U) : \mathcal{F}(U) \to \mathcal{F}(U)$ is precisely the morphism representing multiplication by $e_i \in k[G]$, and hence $\mathcal{F}(U) = \oplus_i \mathrm{im}(\phi_i(U))$ is the standard decomposition of $\mathcal{F}(U)$. The result therefore follows if we can show the image presheaves $\mathrm{im}(\phi_i)$ are already sheaves.

Note that the presheaf $\mathrm{im}(\phi_i)$ is automatically separated, since it is a sub-presheaf of a sheaf. Now suppose $\{U_\alpha\}_\alpha$ is an open cover of an open set $U \subset X$. Suppose $s_\alpha \in \mathcal{F}(U_\alpha)$ are sections such that $\phi_i(U_\alpha)(s_\alpha)|_{U_\alpha \cap U_\beta} = \phi_i(U_\beta)(s_\beta)|_{U_\alpha \cap U_\beta}$ for every α, β. Since $\mathcal{F}$ is a sheaf, there exists $t \in \mathcal{F}(U)$ such that $t|_{U_\alpha} = \phi_i(U_\alpha)(s_\alpha)$ for every α. Then observe that

$$\phi_i(U)(t)|_{U_\alpha} = \phi_i(U_\alpha)(t|_{U_\alpha}) = \phi_i(U_\alpha)(\phi_i(U_\alpha)(s_\alpha)) = \phi_i(U_\alpha)(s_\alpha),$$

since $\phi_i \circ \phi_i = \tilde{\mu}(e_i^2) = \tilde{\mu}(e_i) = \phi_i$. □

Remark 2.2.3. Suppose k is algebraically closed, and suppose the representation ρ_i (corresponding to the component B_i) is one-dimensional. In this case, the associated character χ_i is a group homomorphism. Then we claim $\mathcal{F}_i$ is the χ_i-eigensubsheaf of $\mathcal{F}$; i.e., for every open $U \subset X$,

$$\mathcal{F}_i(U) = \{s \in \mathcal{F}(U) \mid g \cdot s = \chi_i(g)s \quad \forall g \in G\}.$$

Indeed, first suppose $s \in \mathcal{F}(U)$ and $g \cdot s = \chi_i(g)s$ for every $g \in G$. Then

$$\begin{aligned}\phi_i(U)(s) = e_i \cdot s = \frac{1}{|G|}\sum_{g\in G}\chi_i(g^{-1})g \cdot s = \frac{1}{|G|}\sum_{g\in G}\chi_i(g^{-1})\chi_i(g)s \\ = \frac{1}{|G|}\sum_{g\in G}\chi_i(e)s = \frac{1}{|G|}\sum_{g\in G}s \\ = s,\end{aligned}$$

and hence $s \in \operatorname{im}(\phi_i(U)) = \mathcal{F}_i(U)$.

Conversely, suppose $s \in \mathcal{F}_i(U)$, so $s = \phi_i(U)(t)$ for some $t \in \mathcal{F}(U)$. Then for every $g \in G$,

$$\begin{aligned} g \cdot s = g \cdot \phi_i(U)(t) &= g \cdot \left(\frac{1}{|G|} \sum_{h \in G} \chi_i(h^{-1}) h \right) \cdot t \\ &= \frac{1}{|G|} \sum_{h \in G} \chi_i(h^{-1}) gh \cdot t \\ &= \frac{1}{|G|} \sum_{k \in G} \chi_i(k^{-1}g) k \cdot t \\ &= \chi_i(g) \left(\frac{1}{|G|} \sum_{k \in G} \chi_i(k^{-1}) k \right) (t) \\ &= \chi_i(g) \phi_i(U)(t) \\ &= \chi_i(g) s. \end{aligned}$$

In particular, if we assume ρ_1 corresponds to the trivial representation, then $\mathcal{F}_1 = \mathcal{F}^G$ is the subsheaf of G-invariant sections.

It follows that if G is abelian (and k is algebraically closed), then the decomposition $\mathcal{F} = \bigoplus_\chi \mathcal{F}_\chi$ is an eigensheaf decomposition, in which G acts on the sheaf $\mathcal{F}_\chi$ as multiplication by the character χ. It should be noted that in the case $\mathcal{F} = \pi_* \mathcal{O}_Y$ for a morphism $\pi : Y \to X$ from a G-scheme Y, the action of g on $\mathcal{F}$ is induced by the action of g^{-1} on Y, and for this reason some authors (such as Pardini, in [12]) consider $\mathcal{F}_\chi$ to be the χ^{-1}-eigensheaf of $\mathcal{F}$ (or, in the case $\mathcal{F}_\chi$ is invertible, $\mathcal{F}_\chi^{-1}$ is considered the χ-eigensheaf).

Chapter 3

G-covers: A summary

Summary: We recall the definition of a G-cover and outline the general method of constructing such covers. We also briefly investigate their geometry, including branching and ramification.

3.1 Definitions

Definition 3.1.1. Let X be a k-scheme. A G**-cover of** X is a k-scheme Y together with a faithful G-action on Y and a finite morphism $\pi : Y \to X$ which is a geometric quotient for this action.

Remark 3.1.2. For a reference on geometric quotients, see [11]. For our situation, we only need that the property of being a geometric quotient is affine local on the target, and that a morphism of affine schemes, $\operatorname{Spec} B \to \operatorname{Spec} A$, is a geometric quotient if and only if $B^G = A$. Thus, in our case, π is a geometric quotient if and only if $(\pi_* \mathcal{O}_Y)^G = \mathcal{O}_X$.

Proposition 3.1.3. *Suppose X, Y are integral, Noetherian k-schemes and $\pi : Y \to X$ is a flat G-cover. Let $\pi_* \mathcal{O}_Y = \oplus_i \mathcal{F}_i$ be the standard decomposition of $\pi_* \mathcal{O}_Y$.*

Then $\mathcal{F}_i$ is a locally free sheaf of rank equal to the dimension of B_i (over k).

Remark 3.1.4. In the case G is abelian (and k is algebraically closed), it follows the $\mathcal{F}_i$ are all invertible sheaves.

Proof. Suppose $\operatorname{Spec} A \subset X$ is any affine open (and hence A is an integral, Noetherian k-algebra). Since π is a finite, flat morphism, we have $\pi^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$ for some finite, flat A-module B. It follows that B is a free A-module, since any finitely generated flat module over a Noetherian ring is free. Thus, $\pi_*\mathcal{O}_Y$ is locally free. Each $\mathcal{F}_i$ is already coherent, being a direct summand of a coherent sheaf on a Noetherian scheme, and now each is also the direct summand of a finite rank locally free $\mathcal{O}_X$-module. It follows that $(\mathcal{F}_i)_x$ is a direct summand of a finite rank free $\mathcal{O}_{X,x}$-module for every $x \in X$, and hence $(\mathcal{F}_i)_x$ is a projective $\mathcal{O}_{X,x}$-module. Since $\mathcal{O}_{X,x}$ is a local ring, this implies $(\mathcal{F}_i)_x$ is a free $\mathcal{O}_{X,x}$-module [4, pp. 622-4]. Thus, $\mathcal{F}_i$ is a coherent sheaf on a locally Noetherian scheme, with $(\mathcal{F}_i)_x$ a free $\mathcal{O}_{X,x}$-module for every $x \in X$, and hence is locally free [5, pg. 124].

We now calculate the rank of $\mathcal{F}_i$ over the generic point ξ of X. Let E denote the function field of Y. As above, let $\operatorname{Spec} A \subset X$ be an affine open subset, with $\pi^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$ for some finitely generated free A-module B. Then $\pi_*\mathcal{O}_Y|_{\operatorname{Spec} A} = \widetilde{{}^A B}$, where ${}^A B$ denotes B considered as an A-module (rather than as a ring), and so

$$(\pi_*\mathcal{O}_Y)_\xi \cong ({}^A B)_{(0)} = \left\{ \frac{b}{a} \mid b \in B, a \in A^\times \right\}.$$

We plainly have $({}^A B)_{(0)} \subset B_{(0)}$. Conversely, suppose $\frac{b}{b'} \in B_{(0)}$. Since π is a geometric quotient, there is an induced G-action on B with $B^G = A$. Then observe that

$$\frac{b}{b'} = \frac{b \prod\limits_{g \in G \setminus e} g \cdot b'}{\prod\limits_{g \in G} g \cdot b'} = \frac{b''}{a},$$

where $a \in A = B^G$. Thus, $({}^A B)_{(0)} = B_{(0)}$, and hence $(\pi_*\mathcal{O}_Y)_\xi \cong E$.

Now, we plainly have

$$K = A_{(0)} = (B^G)_{(0)} \subset (B_{(0)})^G = E^G.$$

On the other hand, suppose $\frac{b}{b'} \in (B_{(0)})^G$. Since $B_{(0)} = ({}^A B)_{(0)}$, we may assume

$b' = a \in A = B^G$. But then

$$\frac{b}{a} = g \cdot \frac{b}{a} = \frac{g \cdot b}{a}$$

for every $g \in G$, hence $g \cdot b = b$ for every $g \in G$, and hence $b \in B^G = A$.

Thus, E/K is a Galois field extension with group G. By the normal basis theorem, it follows that $E \cong K[G]$ as $K[G]$-modules ([7, pg. 283]). By Proposition 2.2.2, $B = \pi_* \mathcal{O}_Y(\operatorname{Spec} A) = \oplus_i \mathcal{F}_i(\operatorname{Spec} A)$ is the standard decomposition of B. Tensoring with $K[G]$ over $A[G]$ (i.e., localizing at $(0) \subset A$) then gives

$$K[G] \cong E = B_{(0)} = ({}^A B)_{(0)} = \oplus_i (\mathcal{F}_i)_\xi$$

as the standard decomposition of $K[G]$ (as a $k[G]$-module). It follows that $(\mathcal{F}_i)_\xi \cong K[G] \otimes_{k[G]} B_i$, and hence $\dim_K (\mathcal{F}_i)_\xi = \dim_k (B_i)$. □

3.2 Constructing G-covers

Suppose X, Y are integral, Noetherian k-schemes. We have seen that if $\pi : Y \to X$ is a flat G-cover, then $Y \cong \mathbf{Spec}_X \pi_* \mathcal{O}_Y$ and $\pi_* \mathcal{O}_Y = \oplus_i \mathcal{F}_i$, with $\mathcal{F}_i$ a locally free sheaf of rank equal to the dimension of B_i over k. This motivates the following definition.

Definition 3.2.1. Let X be an integral, Noetherian k-scheme. Let $\{\mathcal{F}_i\}_{i=1}^n$ be a collection of locally free $\mathcal{O}_X$-modules, indexed by the ideals in our fixed decomposition $k[G] = \oplus_{i=1}^n B_i$, and assume $\operatorname{rank}(\mathcal{F}_i) = \dim_k(B_i)$ for each i. We call such a collection **building material for a G-cover of X**.

Given building material $\{F_i\}$, let $\mathcal{A} := \oplus_i \mathcal{F}_i$. To construct a G-cover, we must define the appropriate G-action on $\mathcal{A}$, as well as a (commutative, associative) algebra structure on $\mathcal{A}$ compatible with this G-action.

Definition 3.2.2. Given building material $\{F_i\}_i$, we define **building instructions for a G-cover of X** to be a collection of group homomorphisms $\nu_i : G \to \operatorname{Aut}_{\mathcal{O}_X}(\mathcal{F}_i)$ and sheaf morphisms $\mu_{ij} : \mathcal{F}_i \otimes \mathcal{F}_j \to \mathcal{A}$ such that:

(i) the decomposition $\mathcal{A} = \oplus_i \mathcal{F}_i$ is the standard decomposition of $\mathcal{A}$; and

(ii) the μ_{ij} define an algebra structure on $\mathcal{A}$ compatible with this G-action such that $\mathcal{A}^G = \mathcal{O}_X$.

Suppose we are given building material and instructions, as above. Compatibility of the algebra structure with the G-action is the statement that for every $g \in G$ and indices i, j, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_i \otimes \mathcal{F}_j & \xrightarrow{\nu_i(g)\otimes\nu_j(g)} & \mathcal{F}_i \otimes \mathcal{F}_j \\ \downarrow{\scriptstyle \mu_{ij}} & & \downarrow{\scriptstyle \mu_{ij}} \\ \mathcal{A} & \xrightarrow{\oplus_k \nu_k(g)} & \mathcal{A}. \end{array}$$

Remark 3.2.3. If G is an abelian group (and k is algebraically closed), then compatibility with the group action implies factorizations $\mathcal{F}_\chi \otimes \mathcal{F}_{\chi'} \to \mathcal{F}_{\chi\chi'}$. Since the sheaves $\mathcal{F}_\chi$ are invertible, such a morphism is equivalent to a global section of the invertible sheaf $\mathcal{F}_\chi^{-1} \otimes \mathcal{F}_{\chi'}^{-1} \otimes \mathcal{F}_{\chi\chi'}$. This is the situation when G is an abelian group, and one can ultimately show these sections are closely related to the branch divisor of the cover.

Definition 3.2.4. Given building material $\{\mathcal{F}_i\}_i$ and building instructions for a G-cover of X, define $Y := \mathbf{Spec}_X \oplus \mathcal{F}_i$. Then $\pi : Y \to X$ is a finite morphism with $(\pi_*\mathcal{O}_Y)^G = \mathcal{O}_X$, and hence is a geometric quotient. We call $\pi : Y \to X$ the **standard G-cover** associated to the building data, even in the case Y is not integral.

Remark 3.2.5. There is no a priori reason a cover so constructed is flat.

3.3 Ramification and branching

Remark 3.3.1. The results of this section will not be used in subsequent sections, and are presented here only to illustrate some of the local geometry of G-covers.

Throughout this section, assume k is algebraically closed.

Definition 3.3.2. Suppose $\pi : Y \to X$ is a G-cover. For each $g \in G$, define the **g-ramification locus** of π, denoted R_g, as follows. Let $\Gamma_{\nu(g)} : Y \to Y \times_X Y$ denote the graph morphism of $\nu(g)$, and $\Delta : Y \to Y \times_X Y$ the diagonal morphism. Then R_g is defined to be the fiber product

$$\begin{array}{ccc} R_g & \longrightarrow & Y \\ \downarrow & & \downarrow \Gamma_{\nu(g)} \\ Y & \xrightarrow[\Delta]{} & Y \times_X Y \end{array}$$

Remark 3.3.3. Since π is finite, it is separated, and hence Δ is a closed immersion. It follows that $R_g \to Y$ is also a closed immersion, and so we can identify R_g with a closed subscheme of Y. As a set,

$$R_g = \{y \in Y \mid g \cdot y = y\}.$$

Definition 3.3.4. The **(total) ramification locus** of π is the closed subscheme given by the union over all nontrivial g-ramification loci; i.e.,

$$R := \bigcup_{g \in G \setminus e} R_g \subset Y.$$

The **branch locus** of π is the scheme-theoretic image of R under π.

Remark 3.3.5. Since π is finite, it is proper and hence closed. Therefore the set $\pi(R)$ is a closed subset of the integral scheme Y, and D is the reduced induced subscheme structure on the set $\pi(R)$.

Remark 3.3.6. If π is flat, then Zariski's "purity of branch locus" guarantees $R \subset Y$ is a $\mathbb{Q}$-Cartier divisor and $D \subset X$ is a Cartier divisor [15].

Definition 3.3.7. Suppose $\pi : Y \to X$ is a flat G-cover. Let $T \subset R$ be an irreducible component, and let η_T be the generic point of T. The **inertia group** of T is defined to be the subgroup

$$H_T := \{g \in G \mid g \cdot \eta_T = \eta_T\} \leq G.$$

Lemma 3.3.8. *Suppose X, Y are integral, Noetherian k-schemes and $\pi : Y \to X$ is a flat G-cover.*

Then there is an induced faithful G-action on the function field of Y. Moreover, if a subgroup $H \leq G$ fixes a point $y \in Y$, then there is an induced faithful H-action on $\mathcal{O}_{Y,y}$ (as a local $\mathcal{O}_{X,\pi(y)}$-algebra).

Proof. Let η be the generic point of Y, and let ν denote the G-action on Y. Take any $g \in G$. Then $\nu(g^{-1})$ must fix η and hence induce an automorphism $\nu(g^{-1})^\sharp_\eta$ of $\mathcal{O}_{Y,\eta}$ as a local $\mathcal{O}_{X,\pi(y)}$-algebra. This gives a set map $\tilde{\nu} : G \to \mathrm{Aut}_{\mathcal{O}_{X,\pi(y)}}(\mathcal{O}_{Y,\eta})$, which one readily checks is a group homomorphism.

By Examples 2.1.3 and 2.1.5, the faithfulness of ν on Y is equivalent to the faithfulness of $\tilde{\nu}$ on $\pi_*\mathcal{O}_Y$. Thus, if g is nontrivial, there exists some affine open $\operatorname{Spec} A \subset X$ such that $\tilde{\nu}(g)(\operatorname{Spec} A) : \pi_*\mathcal{O}_Y(\operatorname{Spec} A) \to \pi_*\mathcal{O}_Y(\operatorname{Spec} A)$ is nontrivial. By hypothesis, $\pi^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$ for some free A-module B, which is also a ring and an integral domain. Since B is integral, we have an inclusion $B \subset B_{(0)} = \mathcal{O}_{Y,\eta}$ respecting the G-actions; since the action of g on B was nontrivial, it follows that the action of g on $B_{(0)} = \mathcal{O}_{Y,\eta}$ must also be nontrivial.

By the same argument, given any subgroup $H \leq G$ and point $y \in Y$ fixed by H, there is an induced H-action on $\mathcal{O}_{Y,y}$. Let $\operatorname{Spec} A \subset X$ be any affine open neighborhood of $\pi(y)$. Then again $\pi^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$ for some free A-module B, and y is represented by some prime ideal $\mathfrak{p} \subset B$. Since H fixes y, the induced H-action on B preserves $\mathfrak{p}$, and hence induces an H-action on $B_\mathfrak{p}$. Since $(B_\mathfrak{p})_{(0)} = B_{(0)} = \mathcal{O}_{Y,\eta}$, the H-action on $B_\mathfrak{p} = \mathcal{O}_{Y,y}$ induces the faithful H-action on $\mathcal{O}_{Y,\eta}$, and hence must also have been faithful. □

Now suppose $\pi : Y \to X$ is as in the lemma above, and further suppose Y is normal. Let $T \subset R$ be an irreducible component. Then there is an induced faithful action of H_T on $\mathcal{O}_{Y,\eta_T}$. Let $\mathfrak{m} \subset \mathcal{O}_{Y,\eta_T}$ denote the maximal ideal. Since every automorphism of $\mathcal{O}_{Y,\eta_T}$ must preserve $\mathfrak{m}$, we have an induced H_T-action on $\mathfrak{m}$, and hence also on $\mathfrak{m}/\mathfrak{m}^2$. Since η_T is a codimension 1 point of the normal, integral scheme Y, we have $\dim \mathcal{O}_{Y,\eta_T} = \dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$. (Note that, since k is algebraically closed and η_T is a codimension 1 point of Y, the residue field of η_T is necessarily k.) Thus, the H_T-action on the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ gives a one-dimensional k-representation of H_T,

$$\psi_T : H_T \to \mathrm{Gl}_k\left(\mathfrak{m}/\mathfrak{m}^2\right) = k^\times.$$

The following two propositions carry over unchanged from the abelian case [12].

Proposition 3.3.9. *ψ_T is an injection, and hence H_T is cyclic.*

Proof. Since Y is Noetherian and G is a finite group, we have an injection $\mathcal{O}_{Y,\eta_T} \hookrightarrow \hat{\mathcal{O}}_{Y,\eta_T} \cong k[[x]]$ compatible with the H_T-actions. In particular, the H_T-action on $k[[x]]$ is faithful. Now suppose, by way of contradiction, there exists some nontrivial $h \in \ker(\psi_T)$, and so $h \cdot x \equiv x \mod (x)^2$. Since h acts nontrivially on x in $k[[x]]$, we must have $h \cdot x \equiv x + ax^s \mod (x)^{s+1}$ for some $s \geq 2$ and $a \in k^\times$. But then

$$x = h^{|H_T|} \cdot x \equiv x + |H_T|ax^s \mod (x)^{s+1}.$$

Since $a \in k^\times$ and $\mathrm{char}(k) \nmid |H_T|$, this is a contradiction. □

Proposition 3.3.10. *There exists a choice of generator $t \in \mathfrak{m} \subset \mathcal{O}_{Y,\eta_T}$ such that for every $h \in H_T$,*

$$h \cdot t = \psi_T(h)t$$

in $\mathcal{O}_{Y,\eta_T}$.

Proof. By the previous proposition, H_T is cyclic and ψ_T generates the k-characters of H_T. Let $h \in H_T$ be a generator, and let $\zeta := \psi_T(h) \in k^\times$, (so ζ is a primitive $|H_T|^{\text{th}}$ root of unity). By Maschke's theorem, $k[H_T] = \bigoplus_{i=1}^n C_i$, where H_T acts on C_i as ψ_T^i.

Let $t' \in \mathfrak{m}$ be any generator, and let $t' = \sum_i t_i$ be the decomposition induced by the standard decomposition of $\mathfrak{m}$ (as a $k[H_T]$-module). Define x_i to be the image of t_i under the inclusion $\mathcal{O}_{Y,\eta_T} \hookrightarrow \hat{\mathcal{O}}_{Y,\eta_T} = k[[x]]$. For each $x_i \in k[[x]]$, write $x_i \equiv a_i + b_i x \mod (x)^2$ for some $a_i, b_i \in k$. Observe that

$$\begin{aligned} h^s \cdot x = h^s \cdot \sum_i x_i = \sum_i h^s \cdot x_i = \sum_i \left(\psi_T^i(h)\right)^s x_i = \sum_i \zeta^{is} x_i \\ \equiv \sum_i \zeta^{is} a_i + \left(\sum_i \zeta^{is} b_i\right) x \mod (x)^2. \end{aligned}$$

Since h preserves $(x) \subset k[[x]]$, we must have $h^s \cdot x \in (x)$, and hence $\sum_i \zeta^{is} a_i = 0$ for every $s \geq 0$. Thus,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \zeta & \cdots & \zeta^{|H_T|-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \zeta^{|H_T|-1} & \cdots & \zeta^{(|H_T|-1)^2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \cdots \\ a_{|H_T|-1} \end{bmatrix} = \vec{0}.$$

Let L denote the above Vandermonde matrix. Multiplying by the adjoint of L gives $(\det L)\vec{a} = \vec{0}$. Since L is Vandermonde,

$$\det L = \prod_{0 \leq i < j \leq |H_T|-1} (\zeta^i - \zeta^j),$$

which is nonzero (since $\zeta^j = \psi_T(h^j)$, ψ_T is injective, and h generates H_T). Thus, we must have $a_i = 0$ for all i, and hence $x_i \equiv b_i x \mod (x)^2$ for all i.

Now, since h acts on $\mathfrak{m}/\mathfrak{m}^2$ by ζ, the same is true on $(x)/(x)^2$; i.e., $h^s \cdot x \equiv \zeta^s x \mod (x)^2$. This implies $\left(\sum_i \zeta^{is} b_i\right) x \equiv \zeta^s x \mod (x)^2$, and hence $\sum_i \zeta^{is} b_i = \zeta^s$. From this we conclude $\sum_i \zeta^{is}(b_i - \delta_i^1) = 0$ for all s, which implies

$$L \begin{bmatrix} b_0 \\ b_1 - 1 \\ b_2 \\ \cdots \\ b_{|H_T|-1} \end{bmatrix} = \vec{0}.$$

By the previous argument, this implies $b_0 = b_2 = \cdots = b_{|H_T|-1} = 0$ and $b_1 = 1$. Thus, $x_1 \equiv x \mod (x)^2$, and hence $t_1 \equiv t' \mod \mathfrak{m}^2$; i.e., t_1 is the desired generator. □

Now, let $D_0 \subset D$ be an irreducible component of the branch locus, and suppose $T_1, T_2 \subset \pi^{-1}(D_0) \subset R$ are irreducible components, with generic points η_1, η_2, respectively. Since $\pi : Y \to X$ is a geometric quotient, η_1, η_2 lie in the same G-orbit, and hence $\eta_2 = g_0 \cdot \eta_1$ for some $g_0 \in G$. We then have

$$\begin{aligned} H_{T_2} &= \{g \in G \mid g \cdot \eta_2 = \eta_2\} \\ &= \{g \in G \mid g g_0 \cdot \eta_1 = g_0 \cdot \eta_1\} \\ &= \{g \in G \mid g_0^{-1} g g_0 \cdot \eta_1 = \eta_1\} \\ &= g_0 H_{T_1} g_0^{-1}. \end{aligned}$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc} H_{T_1} & \xhookrightarrow{\psi_{T_1}} & k^\times \\ C_{g_0} \downarrow \cong & & \| \\ H_{T_2} & \xhookrightarrow{\psi_{T_2}} & k^\times \end{array},$$

where C_{g_0} is conjugation by g_0. Hence, the induced representations on the corresponding cotangent spaces are isomorphic.

Thus, to every irreducible component $D_0 \subset D$ we can associate the conjugacy class $[H]$ of a cyclic subgroup $H \leq G$, together with a generating character $\psi : H \hookrightarrow k^\times$. For every component $T \subset \pi^{-1}(D_0)$, there is some $g \in G$ such that $H_T = gHg^{-1}$ and $\psi_T = \psi \circ C_{g^{-1}}$.

Chapter 4

S_3-covers

Summary: We first explicitly decompose the ring $k[S_3]$. Using this decomposition, we make a detailed analysis of flat S_3-covers, obtaining a local characterization of such covers. We also examine ramification, as well as the relationship of S_3-covers to triple covers.

4.1 The ring $k[S_3]$

Fix generators $\sigma, \tau \in S_3$, say $\sigma = (1\ 2\ 3), \tau = (2\ 3)$. Write an arbitrary element of $k[S_3]$ as

$$v = a_0e + a_1\sigma + a_2\sigma^2 + a_3\tau + a_4\sigma\tau + a_5\sigma^2\tau,$$

for $a_i \in k$. The (left-) action of S_3 gives

$$\begin{aligned}
\sigma v &= a_2e + a_0\sigma + a_1\sigma^2 + a_5\tau + a_3\sigma\tau + a_4\sigma^2\tau \\
\sigma^2 v &= a_1e + a_2\sigma + a_0\sigma^2 + a_4\tau + a_5\sigma\tau + a_3\sigma^2\tau \\
\tau v &= a_3e + a_5\sigma + a_4\sigma^2 + a_0\tau + a_2\sigma\tau + a_1\sigma^2\tau \\
\sigma\tau v &= a_4e + a_3\sigma + a_5\sigma^2 + a_1\tau + a_0\sigma\tau + a_2\sigma^2\tau \\
\sigma^2\tau v &= a_5e + a_4\sigma + a_3\sigma^2 + a_2\tau + a_1\sigma\tau + a_0\sigma^2\tau.
\end{aligned}$$

Let

$$B_1 = \mathrm{span}_k(e + \sigma + \sigma^2 + \tau + \sigma\tau + \sigma^2\tau),$$

which is the two-sided ideal consisting of all invariant vectors. Similarly, let

$$B_2 = \mathrm{span}_k(e + \sigma + \sigma^2 - \tau - \sigma\tau - \sigma^2\tau),$$

which is the two-sided ideal consisting of all vectors on which S_3 acts as the sign character.

Let B_3 denote the complement of $B_1 \oplus B_2 \subset k[S_3]$. One can readily check B_3 is a two-sided ideal, with a basis given by $\{u_1, u_2, v_1, v_2\}$, where

$$\begin{aligned} u_1 &= -e + \sigma + \tau - \sigma^2\tau \\ u_2 &= -\sigma + \sigma^2 - \tau + \sigma\tau \\ v_1 &= -e + \sigma^2 + \tau - \sigma\tau \\ v_2 &= e - \sigma + \sigma\tau - \sigma^2\tau. \end{aligned}$$

Under the decomposition $k[S_3] = B_1 \oplus B_2 \oplus B_3$, the identity decomposes as $e = e_1 + e_2 + e_3$, where

$$e_1 = \frac{1}{6}(e + \sigma + \sigma^2 + \tau + \sigma\tau + \sigma^2\tau),$$

$$e_2 = \frac{1}{6}(e + \sigma + \sigma^2 - \tau - \sigma\tau - \sigma^2\tau),$$

and

$$e_3 = \frac{1}{3}(2e - \sigma - \sigma^2).$$

4.2 Constructing S_3-covers

Suppose X, Y are integral, Noetherian k-schemes and $\pi : Y \to X$ is a flat G-cover. Then $\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$, where $\mathcal{F}_2$ and $\mathcal{F}_3$ are locally free $\mathcal{O}_X$-modules of ranks 1 and 4, respectively, and where S_3 acts as multiplication by the sign character on $\mathcal{F}_2$, and e_i acts on $\mathcal{F}_3$ as the identity for $i = 3$ and as 0 otherwise.

For notational simplicity, let $\mathcal{A} = \pi_*\mathcal{O}_Y$. The algebra structure on $\mathcal{A}$ is given by an $\mathcal{O}_X$-linear morphism $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A}$, which is equivalent to a morphism

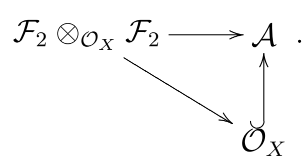
$$(\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X) \oplus (\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F}_2) \oplus (\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{O}_X) \oplus (\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F}_3) \oplus (\mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{O}_X)$$
$$\oplus (\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{F}_2) \oplus (\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{F}_3) \oplus (\mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{F}_2) \oplus (\mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{F}_3) \to A.$$

The first coordinate of this morphism gives the algebra structure on $\mathcal{O}_X$, and the next four coordinates give the left and right $\mathcal{O}_X$-module structures on $\mathcal{F}_2$ and $\mathcal{F}_3$, respectively. It remains to analyze the last four coordinates. Compatibility with the S_3-action implies several factorizations.

Fix any affine open $U \subset X$, and suppose t_1, t_2 are any two sections of $\mathcal{F}_2(U)$. Then for every $g \in S_3$, we must have

$$g(t_1 \otimes t_2) = (g \cdot t_1) \otimes (g \cdot t_2) = (\text{sgn}(g)t_1) \otimes (\text{sgn}(g)t_2) = (\text{sgn}(g))^2(t_1 \otimes t_2) = t_1 \otimes t_2,$$

and so the image of $t_1 \otimes t_2$ must lie in the summand of invariant sections, $\mathcal{O}_X(U)$. Thus, we have a factorization



Similarly, suppose t is any section of $\mathcal{F}_2(U)$ and x is any section of $\mathcal{F}_3(U)$. Then we must have

$$\begin{aligned}
e_3 \cdot (t \otimes x) &= \frac{1}{3}(2e \cdot (t \otimes x) - \sigma \cdot (t \otimes x) - \sigma^2 \cdot (t \otimes x)) \\
&= \frac{1}{3}(t \otimes (2x) - t \otimes (\sigma \cdot x) - t \otimes (\sigma^2 \cdot x)) \\
&= t \otimes (\frac{1}{3}(2x - \sigma \cdot x - \sigma^2 \cdot x)) \\
&= t \otimes (e_3 \cdot x) \\
&= t \otimes x.
\end{aligned}$$

Thus, the image of $t \otimes x$ must be a section of $\mathcal{F}_3(U)$, and hence we must have factorizations

$$\begin{array}{ccc} \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{F}_3 & \longrightarrow & \mathcal{A} \\ & \searrow^{\psi_{23}} & \uparrow \\ & & \mathcal{F}_3 \end{array} , \qquad \begin{array}{ccc} \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{F}_2 & \longrightarrow & \mathcal{A} \\ & \searrow^{\psi_{32}} & \uparrow \\ & & \mathcal{F}_3 \end{array} .$$

Commutativity of the algebra structure implies the following further factorizations:

$$\begin{array}{ccc} \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{F}_2 & \longrightarrow & \mathcal{O}_X \\ \downarrow & \nearrow_{\phi} & \\ S^2\mathcal{F}_2 & & \end{array} , \qquad \begin{array}{ccc} \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{F}_3 & \longrightarrow & \mathcal{A} \\ \downarrow & \nearrow_{\theta} & \\ S^2\mathcal{F}_3 & & \end{array} ,$$

$$\begin{array}{ccc} \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{F}_3 & \xrightarrow{\psi_{23}} & \mathcal{F}_3 \\ \downarrow \cong & \nearrow_{\psi_{32}} & \\ \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{F}_2 & & \end{array}$$

Compatibility with the S_3-action forces many additional relations, however, as does associativity. We examine these relations in the following sections.

4.3 Local analysis

4.3.1 Description of covers: Part I

The aim of this section is to understand the local structure of flat S_3-covers. We first prove the following

Lemma 4.3.1. *Suppose $\pi : Y \to X$ is a flat S_3-cover, and let $\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$ be the standard decomposition. Let $U \subset X$ be any affine open such that $\mathcal{F}_3|_U$ is a free $\mathcal{O}_X|_U$-module.*

Then there exists a basis $\{x_1, x_2, y_1, y_2\}$ for $\mathcal{F}_3(U)$ such that

$$\sigma(x_1) = x_2 \qquad \sigma(x_2) = -x_1 - x_2$$
$$\tau(x_1) = -x_1 \qquad \tau(x_2) = x_1 + x_2$$

and similarly for y_1, y_2.

Proof. Let
$$e_{31} = \frac{1}{3}(e - \sigma^2 - \sigma\tau + \sigma^2\tau)$$
and
$$e_{32} = \frac{1}{3}(e - \sigma + \sigma\tau - \sigma^2\tau).$$
Observe that $e_3 = e_{31} + e_{32}$, $e_{3i}^2 = e_{3i}$ and $e_{31}e_{32} = 0 = e_{32}e_{31}$. We therefore have a direct sum decomposition $\mathcal{F}_3(U) = M_1 \oplus M_2$, where M_i is the image submodule of multiplication by e_{3i}.

Next, observe that

$$\tau e_{32} = \frac{1}{3}(\tau - \sigma^2\tau + \sigma^2 - \sigma) = e_{31}\tau.$$

Thus, $\tau|_{M_2}$ maps M_2 to M_1. Since τ is an order two automorphism of $\mathcal{F}_3(U)$, we thus have that τ isomorphically interchanges the summands M_1 and M_2. In particular, M_1 and M_2 are both rank 2 submodules. Let $\{v_1, v_2\}$ be a basis for M_2. Then $\{\tau\cdot v_1, \tau\cdot v_2\}$ is a basis for M_1, and so $\{v_1, v_2, \tau \cdot v_1, \tau \cdot v_2\}$ is a basis for $\mathcal{F}_3(U)$.

Now define

$$x_1 = v_1 - \tau \cdot v_1$$
$$x_2 = \sigma \cdot v_1 - \sigma\tau \cdot v_1$$
$$y_1 = v_2 - \tau \cdot v_2$$
$$y_2 = \sigma \cdot v_2 - \sigma\tau \cdot v_2.$$

We claim $\{x_1, x_2, y_1, y_2\}$ is the desired basis.

By construction, we have $\sigma \cdot x_1 = x_2$ and $\tau \cdot x_1 = -x_1$, and similarly for y_1, y_2. Next observe that, since e_3 acts as the identity on $\mathcal{F}_3(U)$, we have

$$\frac{1}{3}(2e - \sigma - \sigma^2) \cdot s = s,$$

and hence

$$s + \sigma \cdot s + \sigma^2 \cdot s = 0,$$

for every $s \in \mathcal{F}_3(U)$. In particular, we have

$$\sigma \cdot x_2 = \sigma^2 \cdot x_1 = -x_1 - \sigma \cdot x_1 = -x_1 - x_2,$$
$$\tau \cdot x_2 = \tau\sigma \cdot x_1 = \sigma^2\tau \cdot x_1 = -\sigma^2 \cdot x_1 = x_1 + x_2,$$

and similarly for y_1, y_2.

Thus, it only remains to check the set $\{x_1, x_2, y_1, y_2\}$ is a basis. Observe that

$$\sigma e_{32} = \frac{1}{3}(\sigma - \sigma^2 + \sigma^2\tau - \tau) = -\tau e_{32}.$$

Since e_{32} acts as the identity on M_2, this implies $\sigma \cdot v_i = -\tau \cdot v_i$. Thus, $\sigma\tau \cdot v_i = -\sigma^2 \cdot v_i$, and hence

$$x_2 = \sigma \cdot v_1 - \sigma\tau \cdot v_1 = \sigma \cdot v_1 + \sigma^2 \cdot v_1 = -v_1,$$

and similarly for y_2.

So, the set $\{x_1, x_2, y_1, y_2\}$ is indeed a basis, with inverse change of coordinates given by

$$\begin{aligned} v_1 &= -x_2 \\ v_2 &= -y_2 \\ \tau \cdot v_1 &= -x_1 - x_2 \\ \tau \cdot v_2 &= -y_1 - y_2. \end{aligned}$$

□

Definition 4.3.2. We call any basis satisfying the conclusion of Lemma 4.3.1 a **representation basis** for $\mathcal{F}_3(U)$.

The aim of the remainder of this section is to prove the following

Theorem 4.3.3. *Assume X is an integral, Noetherian k-scheme.*

(a) Suppose $\pi : Y \to X$ is a flat S_3-cover, with Y an integral, Noetherian k-scheme. Let $\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$ be the standard decomposition of $\pi_*\mathcal{O}_Y$. The multiplication in $\mathcal{A} = \pi_*\mathcal{O}_Y$ is determined by a triple of $\mathcal{O}_X$-linear morphisms $\phi : S^2\mathcal{F}_2 \to \mathcal{O}_X, \psi : \mathcal{F}_2 \otimes \mathcal{F}_3 \to \mathcal{F}_3, \theta : S^2\mathcal{F}_3 \to \mathcal{A}$. Let $U = \operatorname{Spec} A \subset X$ be any affine open such that $\mathcal{F}_2|_U$ and $\mathcal{F}_3|_U$ are free $\mathcal{O}_X|_U$-modules. Let t be any generator of $\mathcal{F}_2(U)$.

Then there exists a representation basis $\{x_1, x_2, y_1, y_2\}$ for $\mathcal{F}_3(U)$ such that the morphisms ϕ, ψ, and θ are locally of the form

$$\begin{aligned} \phi(t^2) &= a \\ \psi(t \otimes x_1) &= b(y_1 + 2y_2) \\ \psi(t \otimes x_2) &= -b(2y_1 + y_2) \\ \psi(t \otimes y_1) &= -\frac{a}{3b}(x_1 + 2x_2) \\ \psi(t \otimes y_2) &= \frac{a}{3b}(2x_1 + x_2) \end{aligned}$$

$$
\begin{aligned}
\theta(x_1^2) &= d - f_5(x_1+2x_2) + \frac{3b^2}{a} f_3(y_1+2y_2)\\
\theta(x_1x_2) &= -\frac{d}{2} - f_5(x_1-x_2) + \frac{3b^2}{a} f_3(y_1-y_2)\\
\theta(x_1y_1) &= f_3(x_1+2x_2) + f_5(y_1+2y_2)\\
\theta(x_1y_2) &= \frac{d}{2b}t + f_3(x_1-x_2) + f_5(y_1-y_2)\\
\theta(y_1^2) &= \frac{ad}{3b^2} + \frac{a}{3b^2} f_5(x_1+2x_2) - f_3(y_1+2y_2)\\
\theta(y_1y_2) &= -\frac{ad}{6b^2} + \frac{a}{3b^2} f_5(x_1-x_2) - f_3(y_1-y_2)\\
\theta(x_2^2) &= d + f_5(2x_1+x_2) - \frac{3b^2}{a} f_3(2y_1+y_2)\\
\theta(x_2y_2) &= -f_3(2x_1+x_2) - f_5(2y_1+y_2)\\
\theta(x_2y_1) &= -\frac{d}{2b}t + f_3(x_1-x_2) + f_5(y_1-y_2)\\
\theta(y_2^2) &= \frac{ad}{3b^2} - \frac{a}{3b^2} f_5(2x_1+x_2) + f_3(2y_1+y_2),
\end{aligned}
$$

for some $a, b, f_3, f_5 \in A$, with a, b nonzero, f_3 and f_5 not both zero, and $d = 6(\frac{3b^2}{a} f_3^2 + f_5^2)$.

(b) *Conversely, suppose $\mathcal{F}_2$ and $\mathcal{F}_3$ are locally free $\mathcal{O}_X$-modules of ranks 1 and 4, respectively, and let $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$. Suppose $\mathcal{F}_2, \mathcal{F}_3$ are equipped with S_3-actions such that $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$ is the standard decomposition of $\mathcal{A}$. Suppose $\phi : S^2\mathcal{F}_2 \to \mathcal{O}_X, \psi : \mathcal{F}_2 \otimes \mathcal{F}_3 \to \mathcal{F}_3, \theta : S^2\mathcal{F}_3 \to \mathcal{A}$ are $\mathcal{O}_X$-linear morphisms locally of the form described above.*

Then these morphisms induce an $\mathcal{O}_X$-algebra structure on $\mathcal{A}$ compatible with the S_3-action, and hence define an S_3-cover $\pi : \mathbf{Spec}_X\mathcal{A} \to X$.

We break the proof of this theorem up into three lemmas. For the remainder of this section, assume $\mathcal{F}_2$ and $\mathcal{F}_3$ are locally free $\mathcal{O}_X$-modules of ranks 1 and 4, respectively. Letting $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$, further assume $\mathcal{F}_2, \mathcal{F}_3$ are equipped with S_3-actions such that $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$ is the standard decomposition of $\mathcal{A}$.

For notational simplicity, we shall suppress the notation for ϕ, ψ and θ; e.g., writing simply tx_1 for $\psi(t \otimes x_1)$.

Lemma 4.3.4. *Suppose $U = \operatorname{Spec} A \subset X$ is an affine open such that $\mathcal{F}_2|_U$ and $\mathcal{F}_3|_U$ are free $\mathcal{O}_X|U$-modules.. Let t be any generator for $\mathcal{F}_2(U)$ and $\{x_1, x_2, y_1, y_2\}$ be* **any** *representation basis for $\mathcal{F}_3(U)$. Then every integral algebra structure on $\mathcal{A}$ compatible with the S_3-action is locally of the form*

$$\begin{aligned}
t^2 &= a \\
tx_1 &= b_1(x_1 + 2x_2) + b_3(y_1 + 2y_2) \\
tx_2 &= -b_1(2x_1 + x_2) - b_3(2y_1 + y_2) \\
ty_1 &= c_1(x_1 + 2x_2) - b_1(y_1 + 2y_2) \\
ty_2 &= -c_1(2x_1 + x_2) + b_1(2y_1 + y_2) \\
x_1^2 &= d_1 + d_3(x_1 + 2x_2) + d_5(y_1 + 2y_2) \\
x_1x_2 &= -\frac{1}{2}d_1 + e_2t + e_3x_1 - d_3x_2 + e_5y_1 - d_5y_2 \\
x_1y_1 &= f_1 + f_3(x_1 + 2x_2) + f_5(y_1 + 2y_2) \\
x_1y_2 &= -\frac{1}{2}f_1 + g_2t + g_3x_1 - f_3x_2 + g_5y_1 - f_5y_2 \\
y_1^2 &= h_1 + h_3(x_1 + 2x_2) + h_5(y_1 + 2y_2) \\
y_1y_2 &= -\frac{1}{2}h_1 + j_2t + j_3x_1 - h_3x_2 + j_5y_1 - h_5y_2 \\
x_2^2 &= d_1 - d_3(2x_1 + x_2) - d_5(2y_1 + y_2) \\
x_2y_2 &= f_1 - f_3(2x_1 + x_2) - f_5(2y_1 + y_2) \\
x_2y_1 &= -\frac{1}{2}f_1 - g_2t + f_3x_1 - g_3x_2 + f_5y_1 - g_5y_2 \\
y_2^2 &= h_1 - h_3(2x_1 + x_2) - h_5(2y_1 + y_2),
\end{aligned}$$

for some $a, b_i, c_i, d_i, e_i, f_i, g_i, h_i, j_i \in A$, with $b_1^2 + b_3c_1 = -\frac{a}{3}$.

Proof. Suppose

$$\begin{aligned}
t^2 &= a \\
tx_1 &= b_1x_1 + b_2x_2 + b_3y_1 + b_4y_2 \\
ty_1 &= c_1x_1 + c_2x_2 + c_3y_1 + c_4y_2
\end{aligned}$$

for some $a, b_i, c_i \in A$. Compatibility with the action of σ forces

$$\begin{aligned}
tx_2 = t \cdot \sigma(x_1) &= \sigma(\sigma^2(t) \cdot x_1) = \sigma(tx_1) \\
&= \sigma(b_1x_1 + b_2x_2 + b_3y_1 + b_4y_2) \\
&= b_1\sigma(x_1) + b_2\sigma(x_2) + b_3\sigma(y_1) + b_4\sigma(y_4) \\
&= b_1x_2 + b_2(-x_1 - x_2) + b_3y_2 + b_4(-y_1 - y_2) \\
&= -b_2x_1 + (b_1 - b_2)x_2 - b_4y_1 + (b_3 - b_4)y_2.
\end{aligned}$$

Similarly,

$$ty_2 = -c_2x_1 + (c_1 - c_2)x_2 - c_4y_1 + (c_3 - c_4)y_2.$$

Compatibility with τ forces additional restrictions. We compute

$$\begin{aligned}
\tau(tx_1) &= \tau(b_1x_1 + b_2x_2 + b_3y_1 + b_4y_2) \\
&= -b_1x_1 + b_2(x_1 + x_2) - b_3y_1 + b_4(y_1 + y_2) \\
&= (b_2 - b_1)x_1 + b_2x_2 + (b_4 - b_3)y_2 + b_4y_2 \\
\tau(t) \cdot \tau(x_1) &= (-t) \cdot (-x_1) = tx_1 \\
&= b_1x_1 + b_2x_2 + b_3y_1 + b_4y_2.
\end{aligned}$$

Equating coefficients yields $b_2 = 2b_1, b_4 = 2b_3$. The same calculations for the relation $\tau(ty_1) = \tau(t)\tau(y_1)$ yield $c_2 = 2c_1, c_4 = 2c_3$. Thus, the situation has already simplified to

$$\begin{aligned}
t^2 &= a \\
tx_1 &= b_1(x_1 + 2x_2) + b_3(y_1 + 2y_2) \\
tx_2 &= -b_1(2x_1 + x_2) - b_3(2y_1 + y_2) \\
ty_1 &= c_1(x_1 + 2x_2) + c_3(y_1 + 2y_2) \\
ty_2 &= -c_1(2x_1 + x_2) - c_3(2y_1 + y_2).
\end{aligned}$$

Associativity of the algebra implies several more conditions. We compute

$$\begin{aligned} t \cdot (tx_1) &= t \cdot (b_1x_1 + 2b_1x_2 + b_3y_1 + 2b_3y_2) \\ &= b_1(b_1x_1 + 2b_1x_2 + b_3y_1 + 2b_3y_2) + 2b_1(-2b_1x_1 - b_1x_2 - 2b_3y_1 - b_3y_2) \\ &\quad + b_3(c_1x_1 + 2c_1x_2 + c_3y_1 + 2c_3y_2) + 2b_3(-2c_1x_1 - c_1x_2 - 2c_3y_1 - c_3y_2) \\ &= -3(b_1^2 + b_3c_1)x_1 - 3b_3(b_1 + c_3)y_1. \end{aligned}$$

Since $(t^2) \cdot x_1 = ax_1$, equating coefficients gives

$$-3(b_1^2 + b_3c_1) = a \tag{4.3.5}$$

$$b_3(b_1 + c_3) = 0. \tag{4.3.6}$$

A similar computation for the relation $t \cdot (ty_1) = (t^2) \cdot y_1$ yields

$$c_1(b_1 + c_3) = 0 \tag{4.3.7}$$

$$-3(b_3c_1 + c_3^2) = a. \tag{4.3.8}$$

Note that the relation $t \cdot (tx_2) = (t^2) \cdot x_2$ is now automatically satisfied:

$$\begin{aligned} t \cdot (tx_2) = t \cdot (t\sigma(x_1)) = \sigma(t \cdot tx_1) &= \sigma(t^2 \cdot x_1) \\ t^2 \cdot \sigma(x_1) &= t^2 \cdot x_2. \end{aligned}$$

Similarly, the relation $t \cdot (ty_2) = t^2 \cdot y_2$ is also now satisfied.

Before examining the remaining products, observe that compatibility with σ implies

$$\begin{aligned} x_2^2 &= (\sigma(x_1))^2 = \sigma(x_1^2) \\ x_2y_2 &= \sigma(x_1)\sigma(y_1) = \sigma(x_1y_1) \\ x_2y_1 &= \sigma(x_1)y_1 = \sigma(x_1\sigma^2(y_1)) = -\sigma(x_1y_1 + x_1y_2) \\ y_2^2 &= (\sigma(y_1))^2 = \sigma(y_1^2). \end{aligned}$$

So, it suffices to understand the six products $x_1^2, x_1x_2, x_1y_1, x_1y_2, y_1^2$ and y_1y_2. Suppose

$$\begin{aligned}
x_1^2 &= d_1 + d_2t + d_3x_1 + d_4x_2 + d_5y_1 + d_6y_2 \\
x_1x_2 &= e_1 + e_2t + e_3x_1 + e_4x_2 + e_5y_1 + e_6y_2 \\
x_1y_1 &= f_1 + f_2t + f_3x_1 + f_4x_2 + f_5y_1 + f_6y_2 \\
x_1y_2 &= g_1 + g_2t + g_3x_1 + g_4x_2 + g_5y_1 + g_6y_2 \\
y_1^2 &= h_1 + h_2t + h_3x_1 + h_4x_2 + h_5y_1 + h_6y_2 \\
y_1y_2 &= j_1 + j_2t + j_3x_1 + j_4x_2 + j_5y_1 + j_6y_2,
\end{aligned}$$

for some $d_i, e_i, f_i, g_i, h_i, j_i \in A$. Note that by our previous observation, we then have

$$\begin{aligned}
x_2^2 &= d_1 + d_2t - d_4x_1 + (d_3 - d_4)x_2 - d_6y_1 + (d_5 - d_6)y_2 \\
x_2y_2 &= f_1 + f_2t - f_4x_1 + (f_3 - f_4)x_2 - f_6y_1 + (f_5 - f_6)y_2 \\
x_2y_1 &= -(f_1 + g_1) - (f_2 + g_2)t + (f_4 + g_4)x_1 - (f_3 - f_4 + g_3 - g_4)x_2 \\
&\quad + (f_6 + g_6)y_1 - (f_5 - f_6 + g_5 - g_6)y_2 \\
y_2^2 &= h_1 + h_2t - h_4x_1 + (h_3 - h_4)x_2 - h_6y_1 + (h_5 - h_6)y_2.
\end{aligned}$$

Compatibility with τ imposes many relations. We compute

$$\begin{aligned}
\tau(x_1^2) &= d_1 - d_2t + (d_4 - d_3)x_1 + d_4x_2 + (d_6 - d_5)y_1 + d_6y_2 \\
(\tau(x_1))^2 &= d_1 + d_2t + d_3x_1 + d_4x_2 + d_5y_1 + d_6y_2 \\
\tau(x_1x_2) &= e_1 - e_2t + (e_4 - e_3)x_1 + e_4x_2 + (e_6 - e_5)y_1 + e_6y_2 \\
\tau(x_1)\tau(x_2) &= -(d_1 + e_1) - (d_2 + e_2)t - (d_3 + e_3)x_1 - (d_4 + e_4)x_2 \\
&\quad - (d_5 + e_5)y_1 - (d_6 + e_6)y_2 \\
\tau(x_1y_1) &= f_1 - f_2t + (f_4 - f_3)x_1 + f_4x_2 + (f_6 - f_5)y_1 + f_6y_2 \\
\tau(x_1)\tau(y_1) &= f_1 + f_2t + f_3x_1 + f_4x_2 + f_5y_1 + f_6y_2 \\
\tau(x_1y_2) &= g_1 - g_2t + (g_4 - g_3)x_1 + g_4x_2 + (g_6 - g_5)y_1 + g_6y_2
\end{aligned}$$

$$\begin{aligned}
\tau(x_1)\tau(y_2) &= -(f_1+g_1)-(f_2+g_2)t-(f_3+g_3)x_1-(f_4+g_4)x_2\\
&\quad -(f_5+g_5)y_1-(f_6+g_6)y_2\\
\tau(y_1^2) &= h_1-h_2t+(h_4-h_3)x_1+h_4x_2+(h_6-h_5)y_1+h_6y_2\\
(\tau(y_1))^2 &= h_1+h_2t+h_3x_1+h_4x_2+h_5y_1+h_6y_2\\
\tau(y_1y_2) &= j_1-j_2t+(j_4-j_3)x_1+j_4x_2+(j_6-j_5)y_1+j_6y_2\\
\tau(y_1)\tau(y_2) &= -(h_1+j_1)-(h_2+j_2)t-(h_3+j_3)x_1-(h_4+j_4)x_2\\
&\quad -(h_5+j_5)y_1-(h_6+j_6)y_2.
\end{aligned}$$

Equating coefficients in each pair, we find

$$\begin{aligned}
d_2 = 0, &\quad d_4 = 2d_3, \quad d_6 = 2d_5\\
e_1 = -\frac{1}{2}d_1, &\quad e_4 = -d_3, \quad e_6 = -d_5\\
f_2 = 0, &\quad f_4 = 2f_3, \quad f_6 = 2f_5\\
g_1 = -\frac{1}{2}f_1, &\quad g_4 = -f_3, \quad g_6 = -f_5\\
h_2 = 0, &\quad h_4 = 2h_3, \quad h_6 = 2h_5\\
j_1 = -\frac{1}{2}h_1, &\quad j_4 = -h_3, \quad j_6 = -h_5.
\end{aligned}$$

Lastly, we prove $c_3 = -b_1$. Suppose $c_3 \neq -b_1$. Equations (4.3.4)-(4.3.7) then imply $b_3 = c_1 = 0$ and $b_1^2 = c_3^2 = -\frac{a}{3}$. Since $c_3 \neq -b_1$, the latter equality implies $b_1 = c_3$. The situation is therefore

$$\begin{aligned}
t^2 &= a\\
tx_1 &= b_1(x_1+2x_2)\\
tx_2 &= -b_1(2x_1+x_2)\\
ty_1 &= b_1(y_1+2y_2)\\
ty_2 &= -b_1(2y_1+y_2).
\end{aligned}$$

Integrality of the algebra implies $a, b_1 \neq 0$. Let us now begin analyzing additional associativity conditions. We compute

$$\begin{aligned}
t \cdot x_1^2 &= t \cdot (d_1 + d_3x_1 + 2d_3x_2 + d_5y_1 + 2d_5y_2) \\
&= d_1t + d_3(b_1x_1 + 2b_1x_2) + 2d_3(-2b_1x_1 - b_1x_2) \\
&\quad + d_5(b_1y_1 + 2b_1y_2) + 2d_5(-2b_1y_1 - b_1y_2) \\
&= d_1t - 3b_1d_3x_1 - 3b_1d_5y_1 \\
tx_1 \cdot x_1 &= (b_1x_1 + 2b_1x_2) \cdot x_1 \\
&= b_1(x_1^2 + 2x_1x_2) \\
&= 2b_1e_2t + b_1(d_3 + 2e_3)x_1 + b_1(d_5 + 2e_5)y_1.
\end{aligned}$$

Matching coefficients gives $e_2 = \frac{d_1}{2b_1}, e_3 = -2d_3, e_5 = -2d_5$. It then follows that

$$\begin{aligned}
t \cdot x_1x_2 &= t \cdot \left(-\frac{1}{2}d_1 + \frac{d_1}{2b_1}t - 2d_3x_1 - d_3x_2 - 2d_5y_1 - d_5y_2\right) \\
&= -\frac{1}{2}d_1t + \frac{ad_1}{2b_1} - 2d_3(b_1x_1 + 2b_1x_2) - d_3(-2b_1x_1 - b_1x_2) \\
&- 2d_5(b_1y_1 + 2b_1y_2) - d_5(-2b_1y_1 - b_1y_2) \\
&= \frac{ad_1}{2b_1} - \frac{1}{2}d_1t - 3b_1d_3x_2 - 3b_1d_5y_2 \\
tx_1 \cdot x_2 &= (b_1x_1 + 2b_1x_2) \cdot x_2 \\
&= b_1\left(-\frac{1}{2}d_1 + \frac{d_1}{2b_1}t - 2d_3x_1 - d_3x_2 - 2d_5y_1 - d_5y_2\right) \\
&+ 2b_1(d_1 - 2d_3x_1 - d_3x_2 - 2d_5y_1 - d_5y_2) \\
&= \frac{3b_1d_1}{2} + \frac{1}{2}d_1t - 6b_1d_3x_1 - 3b_1d_3x_2 - 6b_1d_5y_1 - 3b_1d_5y_2.
\end{aligned}$$

Matching coefficients gives $d_1 = d_3 = d_5 = 0$. But this implies $x_1^2 = 0$, which violates the integrality of $\mathcal{A}$. □

We next prove we can change bases to obtain a simpler local form for the algebra structure.

Lemma 4.3.9. *Suppose $U = \operatorname{Spec} A \subset X$ is an affine open such that $\mathcal{F}_2|_U$ and $\mathcal{F}_3|_U$ are free $\mathcal{O}_X|_U$-modules, suppose $\mathcal{A}$ has an integral algebra structure compatible with the S_3-action, and let t be any generator for $\mathcal{F}_2(U)$.*

Then there exists a representation basis $\{x_1', x_2', y_1', y_2'\}$ for $\mathcal{F}_3(U)$ such that the algebra structure is of the form

$$\begin{aligned} t^2 &= a \\ tx_1' &= B(y_1' + 2y_2') \\ ty_1' &= -\frac{a}{3B}(x_1' + 2x_2'), \end{aligned}$$

for some nonzero $a, B \in A$.

Proof. Let $\{x_1, x_2, y_1, y_2\}$ be any representation basis for $\mathcal{F}_3(U)$. By the previous lemma, the algebra structure is then of the form

$$\begin{aligned} t^2 &= a \\ tx_1 &= b_1(x_1 + 2x_2) + b_3(y_1 + 2y_2) \\ tx_2 &= -b_1(2x_1 + x_2) - b_3(2y_1 + y_2) \\ ty_1 &= c_1(x_1 + 2x_2) - b_1(y_1 + 2y_2) \\ ty_2 &= -c_1(2x_1 + x_2) + b_1(2y_1 + y_2), \end{aligned}$$

for some $a, b_1, b_3, c_1 \in A$, with a nonzero, not both b_1, b_3 zero, not both b_1, c_1 zero, and with $b_1^2 + b_3 c_1 = -\frac{a}{3}$.

Suppose $u_1, u_2, v_1, v_2 \in A$ satisfy $u_1 v_2 - u_2 v_1 \neq 0$, and consider the elements

$$\begin{aligned} x_1' &:= u_1 x_1 + u_2 y_1 \\ x_2' &:= u_1 x_2 + u_2 y_2 \\ y_1' &:= v_1 x_1 + v_2 y_1 \\ y_2' &:= v_1 x_2 + v_2 y_2. \end{aligned}$$

We have

$$\begin{vmatrix} u_1 & 0 & v_1 & 0 \\ 0 & u_1 & 0 & v_1 \\ u_2 & 0 & v_2 & 0 \\ 0 & u_2 & 0 & v_2 \end{vmatrix} = (u_1v_2 - u_2v_1)^2 \neq 0,$$

and so $\{x_1', x_2', y_1', y_2'\}$ gives another basis for $\mathcal{F}_3(U)$. For reference, the inverse change of coordinates is

$$\begin{aligned} x_1 &= \frac{1}{u_1v_2 - u_2v_1}(v_2x_1' - u_2y_1') \\ x_2 &= \frac{1}{u_1v_2 - u_2v_1}(v_2x_2' - u_2y_2') \\ y_1 &= \frac{1}{u_1v_2 - u_2v_1}(-v_1x_1' + u_1y_1') \\ y_2 &= \frac{1}{u_1v_2 - u_2v_1}(-v_1x_2' + u_1y_2'). \end{aligned}$$

Observe that

$$\begin{aligned} \sigma(x_1') &= u_1\sigma(x_1) + u_2\sigma(y_1) = u_1x_2 + u_2y_2 = x_2' \\ \sigma(x_2') &= u_1\sigma(x_2) + u_2\sigma(y_2) = u_1(-x_1 - x_2) + u_2(-y_1 - y_2) \\ &= -(u_1x_1 + u_2y_1) - (u_1x_2 + u_2y_2) = -x_1' - x_2' \\ \tau(x_1') &= u_1\tau(x_1) + u_2\tau(y_1) = -u_1x_1 - u_2y_1 = -x_1' \\ \tau(x_2') &= u_1\tau(x_2) + u_2\tau(y_2) = u_1(x_1 + x_2) + u_2(y_1 + y_2) \\ &= (u_1x_1 + u_2y_1) + (u_1x_2 + u_2y_2) = x_1' + x_2', \end{aligned}$$

and similarly for the y_i'. Thus, $\{x_1', x_2', y_1', y_2'\}$ is another representation basis for $\mathcal{F}_3(U)$. We now calculate

$$\begin{aligned} tx_1' &= u_1(tx_1) + u_2(ty_1) \\ &= u_1(b_1x_1 + 2b_1x_2 + b_3y_1 + 2b_3y_2) + u_2(c_1x_1 + 2c_1x_2 - b_1y_1 - 2b_1y_2) \\ &= (u_1b_1 + u_2c_1)(x_1 + 2x_2) + (u_1b_3 - u_2b_1)(y_1 + 2y_2) \end{aligned}$$

$$\begin{aligned}
&= \frac{u_1b_1 + u_2c_1}{u_1v_2 - u_2v_1}(v_2x_1' + 2v_2x_2' - u_2y_1' - 2u_2y_2') \\
&+ \frac{u_1b_3 - u_2b_1}{u_1v_2 - u_2v_1}(-v_1x_1' - 2v_1x_2' + u_1y_1' + 2u_1y_2') \\
&= \frac{1}{u_1v_2 - u_2v_1}(-u_1v_1b_3 + u_1v_2b_1 + u_2v_1b_1 + u_2v_2c_1)(x_1' + 2x_2') \\
&+ \frac{1}{u_1v_2 - u_2v_1}(u_1^2b_3 - 2u_1u_2b_1 - u_2^2c_1)(y_1' + 2y_2').
\end{aligned}$$

A similar calculation yields

$$\begin{aligned}
ty_1' &= \frac{1}{u_1v_2 - u_2v_1}(-v_1^2b_3 + 2v_1v_2b_1 + v_2^2c_1)(x_1' + 2x_2') \\
&+ \frac{1}{u_1v_2 - u_2v_1}(u_1v_1b_3 - u_1v_2b_1 - u_2v_1b_1 - u_2v_2c_1)(y_1' + 2y_2').
\end{aligned}$$

Now, if $b_3 = c_1 = 0$, let $u_1 = u_2 = v_1 = 1$ and $v_2 = -1$. Then $u_1v_2 - u_2v_1 = -2 \neq 0$ and a calculation gives

$$\begin{aligned}
tx_1' &= b_1(y_1' + 2y_2') \\
ty_1' &= b_1(x_1' + 2x_2'),
\end{aligned}$$

and so the lemma holds, with $B = b_1$ (note that $b_1 = -\frac{a}{3b_1}$ in this case).

If $b_3 \neq 0$, let $u_1 = 1, u_2 = 0, v_1 = \frac{b_1}{b_3}$, and $v_2 = 1$. Then $u_1v_2 - u_2v_1 = 1 \neq 0$, and again a calculation gives

$$\begin{aligned}
tx_1' &= b_3(y_1' + 2y_2') \\
ty_1' &= -\frac{a}{3b_3}(x_1' + 2x_2'),
\end{aligned}$$

and the lemma holds, with $B = b_3$.

Finally, if $c_1 \neq 0$, let $u_1 = 0, u_2 = 1, v_1 = 1$, and $v_2 = -\frac{b_1}{c_1}$. Then $u_1v_2 - u_2v_1 = -1 \neq 0$ and a final calculation gives

$$
\begin{aligned}
tx_1' &= c_1(y_1' + 2y_2') \\
ty_1' &= -\frac{a}{3c_1}(x_1' + 2x_2'),
\end{aligned}
$$

and so again the lemma holds, with $B = c_1$. □

We now fully analyze the local form of the algebra structure with respect to this new representation basis, which will complete the proof of the theorem.

Lemma 4.3.10. *Continuing the notation of Lemma 4.3.9, every integral algebra structure on $\mathcal{A}$ compatible with the S_3-action is locally of the form*

$$
\begin{aligned}
t^2 &= a \\
tx_1' &= B(y_1' + 2y_2') \\
tx_2' &= -B(2y_1' + y_2') \\
ty_1' &= -\frac{a}{3B}(x_1' + 2x_2') \\
ty_2' &= \frac{a}{3B}(2x_1' + x_2') \\
x_1'^2 &= D_1 - F_5(x_1' + 2x_2') + \frac{3B^2}{a}F_3(y_1' + 2y_2') \\
x_1'x_2' &= -\frac{1}{2}D_1 - F_5(x_1' - x_2') + \frac{3B^2}{a}F_3(y_1' - y_2') \\
x_1'y_1' &= F_3(x_1' + 2x_2') + F_5(y_1' + 2y_2') \\
x_1'y_2' &= \frac{1}{2B}D_1 t + F_3(x_1' - x_2') + F_5(y_1' - y_2') \\
y_1'^2 &= \frac{a}{3B^2}D_1 + \frac{a}{3B^2}F_5(x_1' + 2x_2') - F_3(y_1' + 2y_2') \\
y_1'y_2' &= -\frac{a}{6B^2}D_1 + \frac{a}{3B^2}F_5(x_1' - x_2') - F_3(y_1' - y_2') \\
x_2'^2 &= D_1 + F_5(2x_1' + x_2') - \frac{3B^2}{a}F_3(2y_1' + y_2') \\
x_2'y_2' &= -F_3(2x_1' + x_2') - F_5(2y_1' + y_2')
\end{aligned}
$$

$$
\begin{aligned}
x'_2y'_1 &= -\frac{1}{2B}D_1t + F_3(x'_1 - x'_2) + F_5(y'_1 - y'_2)\\
y'^2_2 &= \frac{a}{3B^2}D_1 - \frac{a}{3B^2}F_5(2x'_1 + x'_2) + F_3(2y'_1 + y'_2),
\end{aligned}
$$

for some $a, B, F_3, F_5 \in A$, with a, B nonzero, not both F_3 and F_5 zero, and $D_1 = 6(\frac{3B^2}{a}F_3^2 + F_5^2)$.

Conversely, any multiplication structure locally of the above form defines an algebra compatible with the S_3-action.

Proof. Let $\{x'_1, x'_2, y'_1, y'_2\}$ be a representation basis for $\mathcal{F}_3(U)$ as in Lemma 4.3.9. By Lemma 4.3.4, the remaining algebra structure must locally be of the form

$$
\begin{aligned}
x'^2_1 &= D_1 + D_3x'_1 + 2D_3x'_2 + D_5y'_1 + 2D_5y'_2\\
x'_1x'_2 &= -\frac{1}{2}D_1 + E_2t + E_3x'_1 - D_3x'_2 + E_5y'_1 - D_5y'_2\\
x'_1y'_1 &= F_1 + F_3x'_1 + 2F_3x'_2 + F_5y'_1 + 2F_5y'_2\\
x'_1y'_2 &= -\frac{1}{2}F_1 + G_2t + G_3x'_1 - F_3x'_2 + G_5y'_1 - F_5y'_2\\
y'^2_1 &= H_1 + H_3x'_1 + 2H_3x'_2 + H_5y'_1 + 2H_5y'_2\\
y'_1y'_2 &= -\frac{1}{2}H_1 + J_2t + J_3x'_1 - H_3x'_2 + J_5y'_1 - H_5y'_2\\
x'^2_2 &= D_1 - 2D_3x'_1 - D_3x'_2 - 2D_5y'_1 - D_5y'_2\\
x'_2y'_2 &= F_1 - 2F_3x'_1 - F_3x'_2 - 2F_5y'_1 - F_5y'_2\\
x'_2y'_1 &= -\frac{1}{2}F_1 - G_2t + F_3x'_1 - G_3x'_2 + F_5y'_1 - G_5y'_2\\
y'^2_2 &= H_1 - 2H_3x'_1 - H_3x'_2 - 2H_5y'_1 - H_5y'_2,
\end{aligned}
$$

for some $D_i, E_i, F_i, G_i, H_i, J_i \in A$, with not all of any given coefficient type zero. We now begin imposing the remaining associativity relations. We compute

$$
\begin{aligned}
t \cdot x'^2_1 &= t \cdot (D_1 + D_3x'_1 + 2D_3x'_2 + D_5y'_1 + 2D_5y'_2)\\
&= D_1t + D_3(By'_1 + 2By'_2) + 2D_3(-2By'_1 - By'_2)\\
&\quad + D_5\left(-\frac{a}{3B}x'_1 - \frac{2a}{3B}x'_2) + 2D_5(\frac{2a}{3B}x'_1 + \frac{a}{3B}x'_2\right)
\end{aligned}
$$

$$
\begin{aligned}
&= D_1 t + \frac{a}{B} D_5 x_1' - 3BD_3 y_1' \\
tx_1' \cdot x_1' &= (By_1' + 2By_2') \cdot x_1' \\
&= B(F_1 + F_3 x_1' + 2F_3 x_2' + F_5 y_1' + 2F_5 y_2') \\
&\quad + 2B\left(-\frac{1}{2}F_1 + G_2 t + G_3 x_1' - F_3 x_2' + G_5 y_1' - F_5 y_2'\right) \\
&= 2BG_2 t + B(F_3 + 2G_3)x_1' + B(F_5 + 2G_5)y_1'.
\end{aligned}
$$

Matching coefficients gives $G_2 = \frac{1}{2B}D_1, G_3 = \frac{1}{2}(\frac{a}{B^2}D_5 - F_3), G_5 = \frac{1}{2}(-3D_3 - F_5)$.

We next compute

$$
\begin{aligned}
t \cdot x_1' y_1' &= t \cdot (F_1 + F_3 x_1' + 2F_3 x_2' + F_5 y_1' + 2F_5 y_2') \\
&= F_1 t + F_3(By_1' + 2By_2') + 2F_3(-2By_1' - By_2') \\
&\quad + F_5\left(-\frac{a}{3B}x_1' - \frac{2a}{3B}x_2') + 2F_5(\frac{2a}{3B}x_1' + \frac{a}{3B}x_2'\right) \\
&= F_1 t + \frac{a}{B}F_5 x_1' - 3BF_3 y_1' \\
tx_1' \cdot y_1' &= (By_1' + 2By_2') \cdot y_1' \\
&= B(H_1 + H_3 x_1' + 2H_3 x_2' + H_5 y_1' + 2H_5 y_2') \\
&\quad + 2B\left(-\frac{1}{2}H_1 + J_2 t + J_3 x_1' - H_3 x_2' + J_5 y_1' - H_5 y_2'\right) \\
&= 2BJ_2 + B(H_3 + 2J_3)x_1' + B(H_5 + 2J_5)y_1'.
\end{aligned}
$$

Matching coefficients gives $J_2 = \frac{1}{2B}F_1, J_3 = \frac{1}{2}(\frac{a}{B^2}F_5 - H_3), J_5 = \frac{1}{2}(-3F_3 - H_5)$.

Next, we compute

$$
\begin{aligned}
t \cdot y_1'^2 &= t \cdot (H_1 + H_3 x_1' + 2H_3 x_2' + H_5 y_1' + 2H_5 y_2') \\
&= H_1 t + H_3(By_1' + 2By_2') + 2H_3(-2By_1' - By_2') \\
&\quad + H_5\left(-\frac{a}{3B}x_1' - \frac{2a}{3B}x_2'\right) + 2H_5\left(\frac{2a}{3B}x_1' + \frac{a}{3B}x_2'\right) \\
&= H_1 t + \frac{a}{B}H_5 x_1' - 3BH_3 y_1'
\end{aligned}
$$

$$
\begin{aligned}
ty_1' \cdot y_1' &= \left(-\frac{a}{3B}x_1' - \frac{2a}{3B}x_2'\right) \cdot y_1' \\
&= -\frac{a}{3B}(F_1 + F_3x_1' + 2F_3x_2' + F_5y_1' + 2F_5y_2') \\
&\quad - \frac{2a}{3B}\left[-\frac{1}{2}F_1 - \frac{1}{2B}D_1t + F_3x_1' - \frac{1}{2}\left(\frac{a}{B^2}D_5 - F_3\right)x_2'\right. \\
&\quad \left. +F_5y_1' + \frac{1}{2}(3D_3 + F_5)y_2'\right] \\
&= \frac{a}{3B^2}D_1t - \frac{a}{B}F_3x_1' - \frac{a}{B}\left(F_3 - \frac{a}{3B^2}D_5\right)x_2' - \frac{a}{B}F_5y_1' - \frac{a}{B}(F_5 + D_3)y_2'.
\end{aligned}
$$

Matching coefficients gives $H_1 = \frac{a}{3B^2}D_1, H_5 = -F_3, D_5 = \frac{3B^2}{a}F_3, H_3 = \frac{a}{3B^2}F_5$, and $D_3 = -F_5$ (hence also $G_3 = F_3, G_5 = F_5, J_3 = \frac{a}{3B^2}F_5$, and $J_5 = -F_3$).

Continuing, we compute

$$
\begin{aligned}
t \cdot x_1'x_2' &= t \cdot \left(-\frac{1}{2}D_1 + E_2t + E_3x_1' + F_5x_2' + E_5y_1' - \frac{3B^2}{a}F_3y_2'\right) \\
&= -\frac{1}{2}D_1t + aE_2 + E_3(By_1' + 2By_2') + F_5(-2By_1' - By_2') \\
&\quad + E_5\left(-\frac{a}{3B}x_1' - \frac{2a}{3B}x_2'\right) - \frac{3B^2}{a}F_3\left(\frac{2a}{3B}x_1' + \frac{a}{3B}x_2'\right) \\
&= aE_2 - \frac{1}{2}D_1t - \left(\frac{a}{3B}E_5 + 2BF_3\right)x_1' - \left(\frac{2a}{3B}E_5 + BF_3\right)x_2' + B(E_3 - 2F_5)y_1' \\
&\quad + B(2E_3 - F_5)y_2' \\
tx_1' \cdot x_2' &= (By_1' + 2By_2') \cdot x_2' \\
&= B\left(-\frac{1}{2}F_1 - \frac{1}{2B}D_1t + F_3x_1 - F_3x_2' + F_5y_1' - F_5y_2'\right) \\
&\quad + 2B(F_1 - 2F_3x_1' - F_3x_2' - 2F_5y_1' - F_5y_2') \\
&= \frac{3}{2}BF_1 - \frac{1}{2}D_1t - 3BF_3x_1' - 3BF_3x_2' - 3BF_5y_1' - 3BF_5y_2'.
\end{aligned}
$$

Matching coefficients gives $E_2 = \frac{3B}{2a}F_1, E_5 = \frac{3B^2}{a}F_3$, and $E_3 = -F_5$.

We next compute

$$
t \cdot x_1'y_2' = t \cdot \left(-\frac{1}{2}F_1 + \frac{1}{2B}D_1t + F_3x_1' - F_3x_2' + F_5y_1' - F_5y_2'\right)
$$

$$
\begin{aligned}
&= -\frac{1}{2}F_1 t + \frac{a}{2B}D_1 + F_3(By_1' + 2By_2') - F_3(-2By_1' - By_2') \\
&\quad + F_5\left(-\frac{a}{3B}x_1' - \frac{2a}{3B}x_2'\right) - F_5\left(\frac{2a}{3B}x_1' + \frac{a}{3B}x_2'\right) \\
&= \frac{a}{2B}D_1 - \frac{1}{2}F_1 t - \frac{a}{B}F_5 x_1' - \frac{a}{B}F_5 x_2' + 3BF_3 y_1' + 3BF_3 y_2' \\
tx_1' \cdot y_2' &= (By_1' + 2By_2') \cdot y_2' \\
&= B\left(-\frac{a}{6B^2}D_1 + \frac{1}{2B}F_1 t + \frac{a}{3B^2}F_5 x_1' - \frac{a}{3B^2}F_5 x_2' - F_3 y_1' + F_3 y_2'\right) \\
&\quad + 2B\left(\frac{a}{3B^2}D_1 - \frac{2a}{3B^2}F_5 x_1' - \frac{a}{3B^2}F_5 x_2' + 2F_3 y_1' + F_3 y_2'\right) \\
&= \frac{a}{2B}D_1 + \frac{1}{2}F_1 t - \frac{a}{B}F_5 x_1' - \frac{a}{B}F_5 x_2' + 3BF_3 y_1' + 3BF_3 y_2'.
\end{aligned}
$$

Matching coefficients gives $F_1 = 0$, and hence $E_2 = 0 = J_2$.

We next compute

$$
\begin{aligned}
t \cdot y_1' y_2' &= t \cdot \left(-\frac{a}{6B^2}D_1 + \frac{a}{3B^2}F_5 x_1' - \frac{a}{3B^2}F_5 x_2' - F_3 y_1' + F_3 y_2'\right) \\
&= -\frac{a}{6B^2}D_1 t + \frac{a}{3B^2}F_5(By_1' + 2By_2') - \frac{a}{3B^2}F_5(-2By_1' - By_2') \\
&\quad - F_3\left(-\frac{a}{3B}x_1' - \frac{2a}{3B}x_2'\right) + F_3\left(\frac{2a}{3B}x_1' + \frac{a}{3B}x_2'\right) \\
&= -\frac{a}{6B^2}D_1 t + \frac{a}{B}F_3 x_1' + \frac{a}{B}F_3 x_2' + \frac{a}{B}F_5 y_1' + \frac{a}{B}F_5 y_2' \\
ty_1' \cdot y_2' &= \left(-\frac{a}{3B}x_1' - \frac{2a}{3B}x_2'\right) \cdot y_2' \\
&= -\frac{a}{3B}\left(\frac{1}{2B}D_1 t + F_3 x_1' - F_3 x_2' + F_5 y_1' - F_5 y_2'\right) \\
&\quad - \frac{2a}{3B}(-2F_3 x_1' - F_3 x_2' - 2F_5 y_1' - F_5 y_2') \\
&= -\frac{a}{6B^2}D_1 t + \frac{a}{B}F_3 x_1' + \frac{a}{B}F_3 x_2' + \frac{a}{B}F_5 y_1' + \frac{a}{B}F_5 y_2'
\end{aligned}
$$

So, this relation is now automatically satisfied.

Lastly, we compute

$$\begin{aligned}
x_1' \cdot x_1'y_1' &= x_1' \cdot (F_3x_1' + 2F_3x_2' + F_5y_1' + 2F_5y_2') \\
&= F_3\left(D_1 - F_5x_1' - 2F_5x_2' + \frac{3B^2}{a}F_3y_1' + \frac{6B^2}{a}F_3y_2'\right) \\
&\quad + 2F_3\left(-\frac{1}{2}D_1 - F_5x_1' + F_5x_2' + \frac{3B^2}{a}F_3y_1' - \frac{3B^2}{a}F_3y_2'\right) \\
&\quad + F_5(F_3x_1' + 2F_3x_2' + F_5y_1' + 2F_5y_2') \\
&\quad + 2F_5\left(\frac{1}{2B}D_1t + F_3x_1' - F_3x_2' + F_5y_1' - F_5y_2'\right) \\
&= \frac{1}{B}D_1F_5t + \left(\frac{9B^2}{a}F_3^2 + 3F_5^2\right)y_1' \\
x_1'^2 \cdot y_1' &= \left(D_1 - F_5x_1' - 2F_5x_2' + \frac{3B^2}{a}F_3y_1' + \frac{6B^2}{a}F_3y_2'\right) \cdot y_1' \\
&= D_1y_1' - F_5(F_3x_1' + 2F_3x_2' + F_5y_1' + 2F_5y_2') \\
&\quad - 2F_5\left(-\frac{1}{2B}D_1t + F_3x_1' - F_3x_2' + F_5y_1' - F_5y_2'\right) \\
&\quad + \frac{3B^2}{a}F_3\left(\frac{a}{3B^2}D_1 + \frac{a}{3B^2}F_5x_1' + \frac{2a}{3B^2}F_5x_2' - F_3y_1' - 2F_3y_2'\right) \\
&\quad + \frac{6B^2}{a}F_3\left(-\frac{a}{6B^2}D_1 + \frac{a}{3B^2}F_5x_1' - \frac{a}{3B^2}F_5x_2' - F_3y_1' + F_3y_2'\right) \\
&= \frac{1}{B}D_1F_5t + \left(D_1 - \frac{9B^2}{a}F_3^2 - 3F_5^2\right)y_1'
\end{aligned}$$

Matching coefficients gives $D_1 = 6(\frac{3B^2}{a}F_3^2 + F_5^2)$.

After a long and laborious check, one verifies all remaining associativity relations now hold. □

4.3.2 Description of covers: Part II

The previous section succeeded in obtaining a local characterization of flat S_3-covers. The natural next step is to search for a global description. We expect to characterize such covers by a submodule of $\mathrm{Hom}(S^2\mathcal{F}_2, \mathcal{O}_X) \oplus \mathrm{Hom}(\mathcal{F}_2 \otimes \mathcal{F}_3, \mathcal{F}_3) \oplus \mathrm{Hom}(S^2\mathcal{F}_3, \mathcal{A})$, and we would like to find an intrinsic (i.e. basis-free) description of this submodule.

For this task, however, the previous theorem possesses a serious defect. Given two triples of morphisms $(\phi_1, \psi_1, \theta_1), (\phi_2, \psi_2, \theta_2)$ which define S_3-covers, it is not clear their sum also defines an S_3-cover. The problem lies in the fact that the local form of the algebra structure was described with respect to a special choice of representation basis, dependent upon (ϕ, ψ). To remedy this situation, we require a general description of the local form of the algebra structure in terms of *any* representation basis. To that end, the aim of this section is to prove the following

Theorem 4.3.11. *Assume X is an integral, Noetherian k-scheme.*

Suppose $\pi : Y \to X$ is a flat S_3-cover, with Y an integral, Noetherian k-scheme. Let $\mathcal{A} = \pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$ be the standard decomposition of $\pi_*\mathcal{O}_Y$. The multiplication in $\mathcal{A}$ is determined by a triple of $\mathcal{O}_X$-linear morphisms $\phi : S^2\mathcal{F}_2 \to \mathcal{O}_X, \psi : \mathcal{F}_2 \otimes \mathcal{F}_3 \to \mathcal{F}_3, \theta : S^2\mathcal{F}_3 \to \mathcal{A}$. Let $U = \mathrm{Spec}\, A \subset X$ be any affine open such that $\mathcal{F}_2|_U$ and $\mathcal{F}_3|_U$ are free $\mathcal{O}_X|_U$-modules. Let t be any generator for $\mathcal{F}_2(U)$ and $\{x_1, x_2, y_1, y_2\}$ be **any** *representation basis for $\mathcal{F}_3(U)$.*

Then the morphisms ϕ, ψ are locally of the form

$$\begin{aligned}
\phi(t^2) &= a \\
\psi(t \otimes x_1) &= b_1(x_1 + 2x_2) + b_3(y_1 + 2y_2) \\
\psi(t \otimes x_2) &= -b_1(2x_1 + x_2) - b_3(2y_1 + y_2) \\
\psi(t \otimes y_1) &= c_1(x_1 + 2x_2) - b_1(y_1 + y_2) \\
\psi(t \otimes y_2) &= -c_1(2x_1 + x_2) + b_1(2y_1 + y_2)
\end{aligned}$$

for some $a, b_1, b_3, c_1 \in A$, with a nonzero, not both b_1, b_3 zero, not both b_1, c_1 zero, and with $b_1^2 + b_3c_1 = -\frac{a}{3}$. Furthermore, the morphism θ is of one of the following forms:

i) If $b_3 = 0 = c_1$, then θ is of the form

$$\begin{aligned}
\theta(x_1^2) &= d_5(y_1 + 2y_2) \\
\theta(x_1x_2) &= d_5(y_1 - y_2) \\
\theta(x_1y_1) &= 3d_5h_3
\end{aligned}$$

$$\theta(x_1y_2) = -\frac{3}{2}d_5h_3\left(1+\frac{1}{b_1}t\right)$$
$$\theta(y_1^2) = h_3(x_1+2x_2)$$
$$\theta(y_1y_2) = h_3(x_1-x_2)$$
$$\theta(x_2^2) = -d_5(2y_1+y_2)$$
$$\theta(x_2y_2) = 3d_5h_3$$
$$\theta(x_2y_1) = -\frac{3}{2}d_5h_3\left(1-\frac{1}{b_1}t\right)$$
$$\theta(y_2^2) = -h_3(2x_1+x_2),$$

for some nonzero $d_5, h_3 \in A$.

ii) If $b_3 \neq 0$, *then* θ *is of the form*

$$\theta(x_1^2) = D + d_3(x_1+2x_2) + d_5(y_1+2y_2)$$
$$\theta(x_1x_2) = -\frac{1}{2}D + d_3(x_1-x_2) + d_5(y_1-y_2)$$
$$\theta(x_1y_1) = -\frac{b_1}{b_3}D - \frac{1}{b_3}(2b_1d_3+c_1d_5)(x_1+2x_2) - d_3(y_1+2y_2)$$
$$\theta(x_1y_2) = \frac{b_1}{2b_3}D + \frac{1}{2b_3}Dt - \frac{1}{b_3}(2b_1d_3+c_1d_5)(x_1-x_2) - d_3(y_1-y_2)$$
$$\theta(y_1^2) = -\frac{c_1}{b_3}D + \frac{1}{b_3^2}\left(\left(-3b_3c_1-\frac{4a}{3}\right)d_3 + 2b_1c_1d_5\right)(x_1+2x_2) + \frac{1}{b_3}(2b_1d_3+c_1d_5)(y_1+2y_2)$$
$$\theta(y_1y_2) = \frac{c_1}{2b_3}D + \frac{1}{b_3^2}\left(\left(-3b_3c_1-\frac{4a}{3}\right)d_3 + 2b_1c_1d_5\right)(x_1-x_2) + \frac{1}{b_3}(2b_1d_3+c_1d_5)(y_1-y_2)$$
$$\theta(x_2^2) = D - d_3(2x_1+x_2) - d_5(2y_1+y_2)$$
$$\theta(x_2y_2) = -\frac{b_1}{b_3}D + \frac{1}{b_3}(2b_1d_3+c_1d_5)(2x_1+x_2) + d_3(2y_1+y_2)$$
$$\theta(x_2y_1) = \frac{b_1}{2b_3} - \frac{1}{2b_3}Dt - \frac{1}{b_3}(2b_1d_3+c_1d_5)(x_1-x_2) - d_3(y_1-y_2)$$

$$\theta(y_2^2) = -\frac{c_1}{b_3}D - \frac{1}{b_3^2}\left(\left(-3b_3c_1 - \frac{4a}{3}\right)d_3 + 2b_1c_1d_5\right)(2x_1+x_2) - \frac{1}{b_3}(2b_1d_3 + c_1d_5)(2y_1+y_2),$$

for some $d_3, d_5 \in A$, not both zero, and with

$$D = 6d_3^2 - 12\frac{b_1}{b_3}d_3d_5 + \frac{6}{b_3^2}\left(-b_3c_1 + \frac{a}{3}\right)d_5^2.$$

iii) If $c_1 \neq 0$, then θ is of the form

$$\begin{aligned}
\theta(x_1^2) &= -\frac{b_3}{c_1}H + \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(x_1+2x_2) \\
&\quad + \frac{1}{c_1^2}\left(\left(-3b_3c_1 - \frac{4a}{3}\right)h_5 - 2b_1b_3h_3\right)(y_1+2y_2) \\
\theta(x_1x_2) &= \frac{b_3}{2c_1}H + \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(x_1 - x_2) \\
&\quad + \frac{1}{c_1^2}\left(\left(-3b_3c_1 - \frac{4a}{3}\right)h_5 - 2b_1b_3h_3\right)(y_1 - y_2) \\
\theta(x_1y_1) &= \frac{b_1}{c_1}H - h_5(x_1+2x_2) - \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(y_1+2y_2) \\
\theta(x_1y_2) &= -\frac{b_1}{2c_1}H - \frac{1}{2c_1}Ht - h_5(x_1 - x_2) - \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(y_1 - y_2) \\
\theta(y_1^2) &= H + h_3(x_1+2x_2) + h_5(y_1+2y_2) \\
\theta(y_1y_2) &= -\frac{1}{2}H + h_3(x_1 - x_2) + h_5(y_1 - y_2) \\
\theta(x_2^2) &= -\frac{b_3}{c_1}H - \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(2x_1+x_2) \\
&\quad - \frac{1}{c_1^2}\left(\left(-3b_3c_1 - \frac{4a}{3}\right)h_5 - 2b_1b_3h_3\right)(2y_1+y_2) \\
\theta(x_2y_2) &= \frac{b_1}{c_1}H + h_5(2x_1+x_2) + \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(2y_1+y_2) \\
\theta(x_2y_1) &= -\frac{b_1}{2c_1}H + \frac{1}{2c_1}Ht - h_5(x_1 - x_2) - \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(y_1 - y_2) \\
\theta(y_2^2) &= H - h_3(2x_1+x_2) - h_5(2y_1+y_2),
\end{aligned}$$

for some $h_3, h_5 \in A$, not both zero, and with

$$H = 6h_5^2 + 12\frac{b_1}{c_1}h_3h_5 + \frac{6}{c_1^2}\left(-b_3c_1 + \frac{a}{3}\right)h_3^2.$$

Proof. The first statement was already proven in Lemma 4.3.4. We now analyze cases (i)-(iii) successively.

Suppose $b_3 = c_1 = 0$. From the proof of Lemma 4.3.9, we can change coordinates via

$$\begin{aligned}
x_1' &= x_1 + y_1 \\
x_2' &= x_2 + y_2 \\
y_1' &= x_1 - y_1 \\
y_2' &= x_2 - y_2.
\end{aligned}$$

and then apply Lemma 4.3.10 to compute each of the desired products in terms of the F_i. It is useful to first make an intermediate computation, expressing each of the primed products in terms of the unprimed variables. Note that in this case $b_1^2 = -\frac{a}{3}$. We compute

$$\begin{aligned}
(x_1')^2 &= D_1 - (F_3 + F_5)(x_1 + 2x_2) + (F_3 - F_5)(y_1 + 2y_2) \\
x_1'x_2' &= -\frac{1}{2}D_1 - (F_3 + F_5)(x_1 - x_2) + (F_3 - F_5)(y_1 - y_2) \\
x_1'y_1' &= (F_3 + F_5)(x_1 + 2x_2) + (F_3 - F_5)(y_1 + 2y_2) \\
x_1'y_2' &= \frac{1}{2b_1}D_1t + (F_3 + F_5)(x_1 - x_2) + (F_3 - F_5)(y_1 - y_2) \\
(y_1')^2 &= -D_1 - (F_3 + F_5)(x_1 + 2x_2) + (F_3 - F_5)(y_1 + 2y_2) \\
y_1'y_2' &= \frac{1}{2}D_1 - (F_3 + F_5)(x_1 - x_2) + (F_3 - f_5)(y_1 - y_2) \\
(x_2')^2 &= D_1 + (F_3 + F_5)(2x_1 + x_2) - (F_3 - F_5)(2y_1 + y_2) \\
x_2'y_2' &= -(F_3 + F_5)(2x_1 + x_2) - (F_3 - F_5)(2y_1 + y_2) \\
x_2'y_1' &= -\frac{1}{2b_1}D_1t + (F_3 + F_5)(x_1 - x_2) + (F_3 - F_5)(y_1 - y_2)
\end{aligned}$$

$$(y_2')^2 = -D_1 + (F_3 + F_5)(2x_1 + x_2) - (F_3 - F_5)(2y_1 + y_2).$$

We can then compute the desired products, using the inverse change of coordinates

$$\begin{aligned} x_1 &= \frac{1}{2}(x_1' + y_1') \\ x_2 &= \frac{1}{2}(x_2' + y_2') \\ y_1 &= \frac{1}{2}(x_1' - y_1') \\ y_2 &= \frac{1}{2}(x_2' - y_2'). \end{aligned}$$

We compute

$$\begin{aligned} x_1^2 &= \frac{1}{4}((x_1')^2 + 2x_1'y_1' + (y_1')^2) = (F_3 - F_5)(y_1 + 2y_2) \\ x_1x_2 &= \frac{1}{4}(x_1'x_2' + x_1'y_2' + x_2'y_1' + y_1'y_2') = (F_3 - F_5)(y_1 - y_2) \\ x_1y_1 &= \frac{1}{4}((x_1')^2 - (y_1')^2) = \frac{1}{2}D_1 \\ x_1y_2 &= \frac{1}{4}(x_1'x_2' - x_1'y_2' + x_2'y_1' - y_1'y_2') = -\frac{1}{4}D_1\left(1 + \frac{1}{b_1}t\right) \\ y_1^2 &= \frac{1}{4}((x_1')^2 - 2x_1'y_1' + (y_1')^2) = -(F_3 + F_5)(x_1 + 2x_2) \\ y_1y_2 &= \frac{1}{4}(x_1'x_2' - x_1'y_2' - x_2'y_1' + y_1'y_2') = -(F_3 + F_5)(x_1 - x_2). \end{aligned}$$

The remaining products are determined by Lemma 4.3.4. In our original coefficients, Lemma 4.3.4 gave

$$x_1^2 = d_1 + d_3(x_1 + 2x_2) + d_5(y_1 + 2y_2).$$

The above equations thus imply $d_5 = F_3 - F_5, h_3 = -(F_3 + F_5)$, and $f_1 = \frac{1}{2}D_1$. Since

$$D_1 = 6(-F_3^2 + F_5^2) = 6\left(-\left(\frac{1}{2}(d_5 - h_3)\right)^2 + \left(-\frac{1}{2}(d_5 + h_3)\right)^2\right) = 6d_5h_3,$$

the result for case (i) follows.

Next, suppose $b_3 \neq 0$. From the proof of Lemma 4.3.9, we can change coordinates via

$$\begin{aligned}
x_1' &= x_1\\
x_2' &= x_2\\
y_1' &= \frac{b_1}{b_3}x_1 + y_1\\
y_2' &= \frac{b_1}{b_3}x_2 + y_2,
\end{aligned}$$

We then use Lemma 4.3.10 to compute each of the desired products in terms of the F_i. As before, it is useful to first make an intermediate computation, expressing each of the primed products in terms of the unprimed variables. We compute

$$\begin{aligned}
x_1'^2 &= D_1 - F_5(x_1' + 2x_2') + \frac{3b_3^2}{a}F_3(y_1' + 2y_2')\\
&= D_1 + \left(\frac{3b_1b_3}{a}F_3 - F_5\right)(x_1 + 2x_2) + \frac{3b_3^2}{a}F_3(y_1 + 2y_2)\\
x_1'x_2' &= -\frac{1}{2}D_1 - F_5(x_1' - x_2') + \frac{3b_3^2}{a}F_3(y_1' - y_2')\\
&= -\frac{1}{2}D_1 + \left(\frac{3b_1b_3}{a}F_3 - F_5\right)(x_1 - x_2) + \frac{3b_3^2}{a}F_3(y_1 - y_2)\\
x_1'y_1' &= F_3(x_1' + 2x_2') + F_5(y_1' + 2y_2')\\
&= \left(F_3 + \frac{b_1}{b_3}F_5\right)(x_1 + 2x_2) + F_5(y_1 + 2y_2)\\
x_1'y_2' &= \frac{1}{2b_3}D_1t + F_3(x_1' - x_2') + F_5(y_1' - y_2')\\
&= \frac{1}{2b_3}D_1t + \left(F_3 + \frac{b_1}{b_3}F_5\right)(x_1 - x_2) + F_5(y_1 - y_2)\\
y_1'^2 &= \frac{a}{3b_3^2}D_1 + \frac{a}{3b_3^2}F_5(x_1' + 2x_2') - F_3(y_1' + 2y_2')\\
&= \frac{a}{3b_3^2}D_1 - \frac{a}{3b_3^2}\left(\frac{3b_1b_3}{a}F_3 - F_5\right)(x_1 + 2x_2) - F_3(y_1 + 2y_2)\\
y_1'y_2' &= -\frac{a}{6b_3^2}D_1 + \frac{a}{3b_3^2}F_5(x_1' - x_2') - F_3(y_1' - y_2')\\
&= -\frac{a}{6b_3^2}D_1 - \frac{a}{3b_3^2}\left(\frac{3b_1b_3}{a}F_3 - F_5\right)(x_1 - x_2) - F_3(y_1 - y_2)
\end{aligned}$$

$$\begin{aligned}
x_2'^2 &= D_1 + F_5(2x_1' + x_2') - \frac{3b_3^2}{a} F_3(2y_1' + y_2') \\
&= D_1 - \left(\frac{3b_1 b_3}{a} F_3 - F_5 \right)(2x_1 + x_2) - \frac{3b_3^2}{a} F_3(2y_1 + y_2) \\
x_2' y_2' &= -F_3(2x_1' + x_2') - F_5(2y_1' + y_2') \\
&= -\left(F_3 + \frac{b_1}{b_3} F_5 \right)(2x_1 + x_2) - F_5(2y_1 + y_2) \\
x_2' y_1' &= -\frac{1}{2b_3} D_1 t + F_3(x_1' - x_2') + F_5(y_1' - y_2') \\
&= -\frac{1}{2b_3} D_1 t + \left(F_3 + \frac{b_1}{b_3} F_5 \right)(x_1 - x_2) + F_5(y_1 - y_2) \\
y_2'^2 &= \frac{a}{3b_3^2} D_1 - \frac{a}{3b_3^2} F_5(2x_1' + x_2') + F_3(2y_1' + y_2') \\
&= \frac{a}{3b_3^2} D_1 + \frac{a}{3b_3^2} \left(\frac{3b_1 b_3}{a} F_3 - F_5 \right)(2x_1 + x_2) + F_3(2y_1 + y_2)
\end{aligned}$$

It is now straightforward to compute the desired products:

$$\begin{aligned}
x_1^2 = x_1'^2 &= D_1 + \left(\frac{3b_1 b_3}{a} F_3 - F_5 \right)(x_1 + 2x_2) + \frac{3b_3^2}{a} F_3(y_1 + 2y_2) \\
x_1 x_2 = x_1' x_2' &= -\frac{1}{2} D_1 + \left(\frac{3b_1 b_3}{a} F_3 - F_5 \right)(x_1 - x_2) + \frac{3b_3^2}{a} F_3(y_1 - y_2) \\
x_1 y_1 &= x_1' \left(-\frac{b_1}{b_3} x_1' + y_1' \right) = -\frac{b_1}{b_3} x_1'^2 + x_1' y_1' \\
&= -\frac{b_1}{b_3} D_1 + \left(\left(-\frac{3b_1^2}{a} + 1 \right) F_3 + \frac{2b_1}{b_3} F_5 \right)(x_1 + 2x_2) - \left(\frac{3b_1 b_3}{a} F_3 - F_5 \right)(y_1 + 2y_2) \\
x_1 y_2 &= x_1' \left(-\frac{b_1}{b_3} x_2' + y_2' \right) = -\frac{b_1}{b_3} x_1' x_2' + x_1' y_2' \\
&= \frac{b_1}{2b_3} D_1 + \frac{1}{2b_3} D_1 t + \left(\left(-\frac{3b_1^2}{a} + 1 \right) F_3 + \frac{2b_1}{b_3} F_5 \right)(x_1 - x_2) \\
&\quad - \left(\frac{3b_1 b_3}{a} F_3 - F_5 \right)(y_1 - y_2) \\
y_1^2 &= \left(-\frac{b_1}{b_3} x_1' + y_1' \right)^2 = \frac{b_1^2}{b_3^2} x_1'^2 - \frac{2b_1}{b_3} x_1' y_1' + y_1'^2
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{c_1}{b_3} D_1 + \left(\frac{3b_1}{ab_3}(b_1^2 - a)F_3 + \frac{1}{b_3^2}\left(-3b_1^2 + \frac{a}{3}\right) F_5 \right)(x_1 + 2x_2) \\
&\quad + \left(\frac{1}{a}(3b_1^2 - a)F_3 - \frac{2b_1}{b_3} F_5 \right)(y_1 + 2y_2) \\
y_1 y_2 &= \left(-\frac{b_1}{b_3} x_1' + y_1' \right)\left(-\frac{b_1}{b_3} x_2' + y_2' \right) = \frac{b_1^2}{b_3^2} x_1' x_2' - \frac{b_1}{b_3}(x_1' y_2' + x_2' y_1') + y_1' y_2' \\
&= \frac{c_1}{2b_3} D_1 + \left(\frac{3b_1}{ab_3}(b_1^2 - a)F_3 + \frac{1}{b_3^2}\left(-3b_1^2 + \frac{a}{3}\right) F_5 \right)(x_1 - x_2) \\
&\quad + \left(\frac{1}{a}(3b_1^2 - a)F_3 - \frac{2b_1}{b_3} F_5 \right)(y_1 - y_2).
\end{aligned}
$$

From our initial observations, we then automatically have

$$
\begin{aligned}
x_2^2 &= D_1 - \left(\frac{3b_1 b_3}{a} F_3 - F_5 \right)(2x_1 + x_2) - \frac{3b_3^2}{a} F_3 (2y_1 + y_2) \\
x_2 y_2 &= -\frac{b_1}{b_3} D_1 - \left(\left(-\frac{3b_1^2}{a} + 1 \right) F_3 + \frac{2b_1}{b_3} F_5 \right)(2x_1 + x_2) + \left(\frac{3b_1 b_3}{a} F_3 - F_5 \right)(2y_1 + y_2) \\
x_2 y_1 &= \frac{b_1}{2b_3} D_1 + \left(\left(-\frac{3b_1^2}{a} + 1 \right) F_3 + \frac{2b_1}{b_3} F_5 \right)(x_1 - x_2) - \left(\frac{3b_1 b_3}{a} F_3 - F_5 \right)(y_1 - y_2) \\
y_2^2 &= -\frac{c_1}{b_3} D_1 - \frac{1}{2b_3} D_1 t - \left(\frac{3b_1}{ab_3}(b_1^2 - a)F_3 + \frac{1}{b_3^2}\left(-3b_1^2 + \frac{a}{3}\right) F_5 \right)(2x_1 + x_2) \\
&\quad - \left(\frac{1}{a}(3b_1^2 - a)F_3 - \frac{2b_1}{b_3} F_5 \right)(2y_1 + y_2).
\end{aligned}
$$

Comparing with our original coefficients, we thus have $d_1 = D_1, d_3 = \frac{3b_1 b_3}{a} F_3 - F_5, d_5 = \frac{3b_3^2}{a} F_3$, and hence $F_3 = \frac{a}{3b_3^2} d_5, F_5 = \frac{b_1}{b_3} d_5 - d_3$. Substituting these into our equations and simplifying gives the result.

Lastly, suppose $c_1 \neq 0$. If we change coordinates via $\tilde{x}_i = y_i, \tilde{y}_i = x_i$, then $\{\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2\}$ is another representation basis for $\mathcal{F}_3$ and

$$
\begin{aligned}
t\tilde{x}_1 &= \tilde{b}_1(\tilde{x}_1 + 2\tilde{x}_2) + \tilde{b}_3(\tilde{y}_1 + 2\tilde{y}_2) \\
t\tilde{y}_1 &= \tilde{c}_1(\tilde{x}_1 + 2\tilde{x}_2) - \tilde{b}_1(\tilde{y}_1 + 2\tilde{y}_2),
\end{aligned}
$$

where $\tilde{b}_1 = -b_1, \tilde{b}_3 = c_1 \neq 0$, and $\tilde{c}_1 = b_3$. Also, if

$$\begin{aligned}\tilde{x}_1^2 &= \tilde{D} + \tilde{d}_1(\tilde{x}_1 + 2\tilde{x}_2) + \tilde{d}_5(\tilde{y}_1 + 2\tilde{y}_2) \\ &= \tilde{D} + \tilde{d}_5(x_1 + 2x_2) + \tilde{d}_3(y_1 + 2y_2),\end{aligned}$$

then since

$$\tilde{x}_1^2 = y_1^2 = H + h_3(x_1 + 2x_2) + h_5(y_1 + 2y_2),$$

we have $\tilde{D} = H, \tilde{d}_3 = h_5$, and $\tilde{d}_5 = h_3$. Thus, the previous case immediately gives

$$\begin{aligned}
\tilde{x}_1^2 &= H + h_5(\tilde{x}_1 + 2\tilde{x}_2) + h_3(\tilde{y}_1 + 2\tilde{y}_2) \\
\tilde{x}_1\tilde{x}_2 &= -\frac{1}{2}H + h_5(\tilde{x}_1 - \tilde{x}_2) + h_3(\tilde{y}_1 - \tilde{y}_2) \\
\tilde{x}_1\tilde{y}_1 &= \frac{b_1}{c_1}H - \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(\tilde{x}_1 + 2\tilde{x}_2) - h_5(\tilde{y}_1 + 2\tilde{y}_2) \\
\tilde{x}_1\tilde{y}_2 &= -\frac{b_1}{2c_1}H + \frac{1}{2c_1}Ht - \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(\tilde{x}_1 - \tilde{x}_2) - h_5(\tilde{y}_1 - \tilde{y}_2) \\
\tilde{y}_1^2 &= -\frac{b_3}{c_1}H + \frac{1}{c_1^2}\left(\left(-3b_3c_1 - \frac{4a}{3}\right)h_5 - 2b_1b_3h_3\right)(\tilde{x}_1 + 2\tilde{x}_2) \\
&\quad + \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(\tilde{y}_1 + 2\tilde{y}_2) \\
\tilde{y}_1\tilde{y}_2 &= \frac{b_3}{2c_1}H + \frac{1}{c_1^2}\left(\left(-3b_3c_1 - \frac{4a}{3}\right)h_5 - 2b_1b_3h_3\right)(\tilde{x}_1 - \tilde{x}_2) \\
&\quad + \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(\tilde{y}_1 - \tilde{y}_2) \\
\tilde{x}_2^2 &= H - h_5(2\tilde{x}_1 + \tilde{x}_2) - h_3(2\tilde{y}_1 + \tilde{y}_2) \\
\tilde{x}_2\tilde{y}_2 &= \frac{b_1}{c_1}H + \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(2\tilde{x}_1 + \tilde{x}_2) + h_5(2\tilde{y}_1 + \tilde{y}_2) \\
\tilde{x}_2\tilde{y}_1 &= -\frac{b_1}{2c_1}H - \frac{1}{2c_1}Ht - \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(\tilde{x}_1 - \tilde{x}_2) - h_5(\tilde{y}_1 - \tilde{y}_2) \\
\tilde{y}_2^2 &= -\frac{b_3}{c_1}H - \frac{1}{c_1^2}\left(\left(-3b_3c_1 - \frac{4a}{3}\right)h_5 - 2b_1b_3h_3\right)(2\tilde{x}_1 + \tilde{x}_2) \\
&\quad - \frac{1}{c_1}(-2b_1h_5 + b_3h_3)(2\tilde{y}_1 + \tilde{y}_2),
\end{aligned}$$

with

$$h_1 = 6h_5^2 + 12\frac{b_1}{c_1}h_3h_5 + \frac{6}{c_1^2}\left(-b_3c_1 + \frac{a}{3}\right)h_3^2.$$

The result follows.

□

4.4 Ramification

Suppose $\pi : Y \to X$ is a flat S_3-cover, where X, Y are integral, Noetherian k-schemes. Let $\mathcal{A} = \pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$ be the standard decomposition of $\pi_*\mathcal{O}_Y$. Let $U = \operatorname{Spec} A \subset X$ be any affine open such that $\mathcal{F}_2|_U$ and $\mathcal{F}_3|_U$ are free $\mathcal{O}_X|_U$-modules. Let t be any local generator for $\mathcal{F}_2(U))$ and $\{x_1, x_2, y_1, y_2\}$ be a local representation basis for $\mathcal{F}_3(U)$ as in Theorem 4.3.3. According to Theorem 4.3.3, if we consider t, x_1, x_2, y_1, y_2 as variables over A, and let

$$\begin{aligned}
F(t, x_1, x_2, y_1, y_2) &= t^2 - a \\
G_1(t, x_1, x_2, y_1, y_2) &= tx_1 - b(y_1 + 2y_2) \\
G_2(t, x_1, x_2, y_1, y_2) &= tx_2 + b(2y_1 + y_2) \\
G_3(t, x_1, x_2, y_1, y_2) &= ty_1 + \frac{a}{3b}(x_1 + 2x_2) \\
G_4(t, x_1, x_2, y_1, y_2) &= ty_2 - \frac{a}{3b}(2x_1 + x_2) \\
H_1(t, x_1, x_2, y_1, y_2) &= x_1^2 - d + f_5(x_1 + 2x_2) - \frac{3b^2}{a}f_3(y_1 + 2y_2) \\
H_2(t, x_1, x_2, y_1, y_2) &= x_1x_2 + \frac{d}{2} + f_5(x_1 - x_2) - \frac{3b^2}{a}f_3(y_1 - y_2) \\
H_3(t, x_1, x_2, y_1, y_2) &= x_1y_1 - f_3(x_1 + 2x_2) - f_5(y_1 + 2y_2) \\
H_4(t, x_1, x_2, y_1, y_2) &= x_1y_2 - \frac{d}{2b}t - f_3(x_1 - x_2) - f_5(y_1 - y_2) \\
H_5(t, x_1, x_2, y_1, y_2) &= y_1^2 - \frac{ad}{3b^2} - \frac{a}{3b^2}f_5(x_1 + 2x_2) + f_3(y_1 + 2y_2) \\
H_6(t, x_1, x_2, y_1, y_2) &= y_1y_2 + \frac{ad}{6b^2} - \frac{a}{3b^2}f_5(x_1 - x_2) + f_3(y_1 - y_2) \\
H_7(t, x_1, x_2, y_1, y_2) &= x_2^2 - d - f_5(2x_1 + x_2) + \frac{3b^2}{a}f_3(2y_1 + y_2)
\end{aligned}$$

$$H_8(t,x_1,x_2,y_1,y_2) = x_2y_2 + f_3(2x_1+x_2) + f_5(2y_1+y_2)$$
$$H_9(t,x_1,x_2,y_1,y_2) = x_2y_1 + \frac{d}{2b}t - f_3(x_1-x_2) - f_5(y_1-y_2)$$
$$H_{10}(t,x_1,x_2,y_1,y_2) = y_2^2 - \frac{ad}{3b^2} + \frac{a}{3b^2}f_5(2x_1+x_2) - f_3(2y_1+y_2),$$

then $\mathcal{A}(U)$ can be viewed as the quotient ring

$$B = \frac{A[t,x_1,x_2,y_1,y_2]}{(F,G_1,\ldots,G_4,H_1,\ldots,H_{10})}.$$

Thus, if $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, then this expresses Y as subvariety of $\mathbb{A}^5_X$. The ramification locus is then the locus where the Jacobian matrix of $F, G_1, \ldots, G_4$, $H_1, \ldots, H_{10}$ with respect to the variables t, x_1, x_2, y_1, y_2 does not have maximal rank. This matrix is

$$\begin{bmatrix}
2t & 0 & 0 & 0 & 0 \\
x_1 & t & 0 & -b & -2b \\
x_2 & 0 & t & 2b & b \\
y_1 & \frac{a}{3b} & \frac{2a}{3b} & t & 0 \\
y_2 & -\frac{2a}{3b} & -\frac{a}{3b} & 0 & t \\
0 & 2x_1+f_5 & 2f_5 & -\frac{3b^2}{a}f_3 & -\frac{6b^2}{a}f_3 \\
0 & y_1-f_3 & -2f_3 & x_1-f_5 & -2f_5 \\
-\frac{1}{2b} & y_2-f_3 & f_3 & -f_5 & x_1+f_5 \\
0 & -\frac{a}{3b^2}f_5 & -\frac{2a}{3b^2}f_5 & 2y_1+f_3 & 2f_3 \\
0 & -\frac{a}{3b^2}f_5 & \frac{a}{3b^2}f_5 & y_2+f_3 & y_1-f_3 \\
0 & -2f_5 & 2x_2-f_5 & \frac{6b^2}{a}f_3 & \frac{3b^2}{a}f_3 \\
0 & 2f_3 & y_2+f_3 & 2f_5 & x_2+5 \\
\frac{d}{2b} & -f_3 & y_1+f_3 & x_2-f_5 & f_5 \\
0 & \frac{2a}{3b^2}f_5 & \frac{a}{3b^2}f_5 & -2f_3 & 2y_2-f_3
\end{bmatrix}.$$

This matrix has $\binom{15}{5} = 3003$ minors of size 5×5, and the ramification locus is defined by the ideal $R \subset B$ generated by these 3003 elements.

4.5 Relation to triple covers

Triple covers have been well studied by Miranda in [9], and are closely related to S_3-covers. We can use the theory of triple covers as a check for our previous calculations. Recall the following

Theorem 4.5.1. *[9, Thm. 2.7] Assume X is an integral, Noetherian k-scheme.*

(a) Let $f : Z \to X$ be a triple cover. Then $\mathcal{B} = f_*\mathcal{O}_Z$ is a flat, rank 3 locally free sheaf of $\mathcal{O}_X$-algebras, and $\mathcal{B} \cong \mathcal{O}_X \oplus \mathcal{E}$, where $\mathcal{E}$ is the locally free rank two $\mathcal{O}_X$-submodule of $\mathcal{B}$ known as the Tschirnhausen module. The multiplication in $\mathcal{B}$ is determined by an $\mathcal{O}_X$-linear morphism $\phi : S^2\mathcal{E} \to \mathcal{O}_X \oplus \mathcal{E}$. Let $U = \operatorname{Spec} A \subset X$ be an affine open such that $\mathcal{E}|_U$ is a free $\mathcal{O}_X|_U$-module. If $\{z, w\}$ is any basis for $\mathcal{E}(U)$, then ϕ is locally of the form

$$\begin{aligned}
\phi(z^2) &= 2(\tilde{a}^2 - \tilde{b}\tilde{d}) + \tilde{a}z + \tilde{b}w \\
\phi(zw) &= -(\tilde{a}\tilde{d} - \tilde{b}\tilde{c}) - \tilde{d}z - \tilde{a}w \\
\phi(w^2) &= 2(\tilde{d}^2 - \tilde{a}\tilde{c}) + \tilde{c}z + \tilde{d}w,
\end{aligned}$$

for some $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in A$, with $\tilde{b}, \tilde{c}$ nonzero.

(b) Conversely, suppose $\mathcal{E}$ is a rank two locally free $\mathcal{O}_X$-module. Let $U = \operatorname{Spec} A \subset X$ be an affine open such that $\mathcal{E}|_U$ is a free $\mathcal{O}_X|_U$-module, and let $\{z, w\}$ be any basis for $\mathcal{E}(U)$. Suppose $\phi : S^2\mathcal{E} \to \mathcal{O}_X \oplus \mathcal{E}$ is a morphism locally of the form above.

Then ϕ induces an $\mathcal{O}_X$-algebra structure on $\mathcal{B} = \mathcal{O}_X \oplus \mathcal{E}$, making $f : \mathbf{Spec}_X\mathcal{B} \to X$ a triple cover.

Suppose we are given a flat S_3-cover $\pi : Y \to X$, with X, Y integral, Noetherian k-schemes. Let $\mathcal{A} = \pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$ be the standard decomposition. Let $U = \operatorname{Spec} A \subset X$ be any affine subset such that $\mathcal{F}_2|_U$ and $\mathcal{F}_3|_U$ are free $\mathcal{O}_X|_U$-modules. Let t be a generator of $\mathcal{F}_2(U)$ and $\{x_1, x_2, y_1, y_2\}$ be a representation basis for $\mathcal{F}_3(U)$ as in Theorem 4.3.3.

We claim $\pi' : \mathbf{Spec}_X \mathcal{A}^\tau \to X$ is a triple cover. To prove this, observe that for an arbitrary section $s = a_0 + a_1 t + a_2 x_1 + a_3 x_2 + a_4 y_1 + a_5 y_2 \in \mathcal{A}(U)$, we have

$$\tau(s) = a_0 - a_1 t + (a_3 - a_2)x_1 + a_3 x_2 + (a_5 - a_4)y_1 + a_5 y_2.$$

In particular, s is fixed by τ if and only if $a_3 = 2a_2$ and $a_5 = 2a_4$. Thus, $\mathcal{A}^\tau = \mathcal{O}_X \oplus \mathcal{E}$, where $\mathcal{E}$ is a locally free rank 2 module with local basis $\{z, w\}$, where $z = x_1 + 2x_2, w = y_1 + 2y_2$. Note that $\mathcal{A}^\tau$ is a subalgebra, with algebra structure locally given by

$$\begin{aligned}
z^2 &= x_1^4 x_1 x_2 + 4x_2^2 \\
&= 3d + 3f_5 z - \frac{9b^2}{a} f_3 w \\
zw &= x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 4x_2 y_2 \\
&= -3f_3 z - 3f_5 w \\
w^2 &= y_1^2 + 4y_1 y_2 + 4y_2^2 \\
&= \frac{ad}{b^2} - \frac{a}{b^2} f_5 z + 3f_3 w.
\end{aligned}$$

If we let $\tilde{a} = 3f_5, \tilde{b} = -\frac{9b^2}{a} f_3, \tilde{c} = -\frac{a}{3b^3} f_5, \tilde{d} = 3f_3$, then we have

$$\begin{aligned}
2(\tilde{a} - \tilde{b}\tilde{d}) &= 18 \left(\frac{3b^2}{a} f_3^2 + f_5^2 \right) = 3d \\
-(\tilde{a}\tilde{d} - \tilde{b}\tilde{c}) &= -(9f_3 f_5 - 9f_3 f_5) = 0 \\
2(\tilde{d}^2 - \tilde{a}\tilde{c}) &= \frac{6a}{b^2} \left(\frac{3b^2}{a} f_3^2 + f_5^2 \right) = \frac{a}{b^2} d.
\end{aligned}$$

We can therefore rewrite the local algebra structure on $\mathcal{A}^\tau$ as

$$\begin{aligned}
z^2 &= 2(\tilde{a}^2 - \tilde{b}\tilde{d}) + \tilde{a}z + \tilde{b}w \\
zw &= -(\tilde{a}\tilde{d} - \tilde{b}\tilde{c}) - \tilde{d}z - \tilde{a}w \\
w^2 &= 2(\tilde{d}^2 - \tilde{a}\tilde{c}) + \tilde{c}z + \tilde{d}w.
\end{aligned}$$

By Miranda's theorem, $\mathbf{Spec}_X \mathcal{A}^\tau \to X$ is therefore a triple cover, as claimed.

Of course, we could have also considered the subalgebras $\mathcal{A}^{\sigma\tau}$ and $\mathcal{A}^{\sigma^2\tau}$, defined by the $\sigma\tau$- and $\sigma^2\tau$-invariant sections, respectively. However, if s is a τ-invariant section of $\mathcal{A}$, then

$$\sigma\tau \cdot (\sigma^2 s) = \sigma^2\tau \cdot s = \sigma^2 s,$$

and so $\sigma^2 s$ is a $\sigma\tau$-invariant section. Conversely, if t is $\sigma\tau$-invariant section, then

$$\tau \cdot (\sigma t) = \sigma^2\tau \cdot t = \sigma \cdot (\sigma\tau t) = \sigma t,$$

and so σt is a τ-invariant section. Thus, $\mathcal{A}^{\sigma\tau} = \sigma^2(\mathcal{A}^\tau)$, and hence $\mathbf{Spec}_X\mathcal{A}^{\sigma\tau} \to X$ is conjugate to the triple cover $\mathbf{Spec}_X\mathcal{A}^\tau \to X$. Similarly, we have $\mathcal{A}^{\sigma^2\tau} = \sigma(\mathcal{A}^\tau)$, and so $\mathbf{Spec}_X\mathcal{A}^{\sigma^2\tau} \to X$ is also conjugate to the triple cover $\mathbf{Spec}_X\mathcal{A}^\tau \to X$.

Chapter 5

Future work

With the local structure of flat S_3-covers understood in detail, many avenues for investigation are now open. We briefly outline a few such projects.

Global description. As noted previously, the next natural step in understanding S_3-covers is to obtain a global description. If one could obtain a uniform version of Theorem 4.3.11, it would then be possible to define a submodule of $\mathrm{Hom}(S^2\mathcal{F}_2, \mathcal{O}_X) \oplus \mathrm{Hom}(\mathcal{F}_2 \otimes \mathcal{F}_3, \mathcal{F}_3) \oplus \mathrm{Hom}(S^2\mathcal{F}_3, \mathcal{A})$ consisting of triples of morphisms (ϕ, ψ, θ) which define S_3-covers. Using the local description as a guide, one would like to obtain a basis-free characterization of this submodule. This strategy closely parallels the technique used by Miranda in [9] to obtain a global description of triple covers, in which the submodule of $\mathrm{Hom}(S^2\mathcal{E}, \mathcal{E})$ consisting of triple cover homomorphisms is identified with $\mathrm{Hom}(S^2\mathcal{E}, \bigwedge^2 \mathcal{E})$.

Singularities. With the full local description of S_3-covers at our disposal, it should be possible to directly compute the singularities of a given S_3-cover. In fact, a general statement for arbitrary finite groups G may be possible. Suppose $\pi : Y \to X$ is a G-cover of a nonsingular scheme X. As the morphism π is étale away from the ramification locus $R \subset Y$, we can focus attention locally around $R \subset Y$ and $D \subset X$. In the case of abelian covers, the locus of singularities is described by the following

Theorem 5.0.1. *([12, Prop. 3.1]) Assume X, Y are integral, Noetherian k-schemes, and $\pi : Y \to X$ is a flat G-cover. Further assume X is nonsingular.*

Then Y is nonsingular over a point $x \in X$ if and only if one of the following conditions holds:

i) x is not a branch point of π; or,

ii) x belongs only to one component Δ of D and x is a smooth point of Δ; or,

iii) x lies on components $D_{H_1,\psi_1}, \ldots, D_{H_r,\psi_r}$ of D and:

a) the map $H_1 \oplus \cdots \oplus H_r \to G$ is an injection; and

b) if b_i is a local equation for D_{H_i,ψ_i} around x, $i = 1, \ldots, r$, then the subspace of $\mathfrak{m}/\mathfrak{m}^2$ generated by $db_1, \ldots, db_r$ has dimension r.

This theorem is proved by localizing the cover around the branch divisor $D \subset X$, and then factoring the cover as a totally ramified H-cover followed by an étale map, where H is the inertia subgroup associated to D. Since the inertia subgroups are all cyclic, this reduces the question to that of a cyclic cover. We have seen, however, that the inertia subgroups are cyclic even in the case of a general, nonabelian group. It seems likely, then, that a similar description of the singularities will hold in this more general setting.

Deformations. Since the construction of S_3-covers essentially amounts to the definition of an algebra structure on a given locally free sheaf $\mathcal{A}$, one might expect deformations of such covers to correspond to deformations of this algebra structure. This is indeed the case for abelian covers [12]. Understanding the deformation theory of S_3-covers is very useful for understanding moduli spaces containing such covers.

For example, in [3], branched abelian covers are used to investigate the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on surfaces defined over $\bar{\mathbb{Q}}$. Using an explicit correspondence between the deformations of an abelian cover and its branch locus, one is able to produce, for every nontrivial element of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, a pair of conjugate surfaces which are not locally deformation equivalent. In doing so, one thus establishes that $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the irreducible components of the moduli space of surfaces over $\bar{\mathbb{Q}}$.

Further relationships with triple covers. We have seen how, from an S_3-cover $\pi : Y \to X$, one produces a triple cover $\pi' : \mathbf{Spec}_X(\pi_*\mathcal{O}_Y)^\tau \to X$. Conversely, if $f : Z \to X$ is a triple cover, and if we let Y denote the closure of $(Z \times_X Z)\backslash\Delta \subset Z \times_X Z$, then Y should represent the Galois normalization of $f : Z \to Y$, and hence define an S_3-cover. With the local nature of both triple covers and S_3-covers explicitly described, it should be possible to verify this claim. One could then translate statements about triple covers to statements about certain S_3-covers, and conversely.

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