The Mathematics of Doodling

University of Utah
Teachers' Math Circle

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Part I: Circular Reasoning

Draw a random figure below, and then trace a curve around it, as closely as you can. Then trace another curve. And another. Repeat a bunch of times.

Did the traced curves seem to get more and more circular?

Definition: Given a plane set X and number $r \geq 0$, let

$$N_r(X) = \{y \mid |y - x| \leq r \text{ for some } x \in X\}$$

Conjecture (version 1): In some sense, does (the boundary of)

$$N_r(N_r(\cdots(N_r(X))\cdots))$$

become ever more circular as we iterate N_r?

Related Question: How does our choice of r affect things?

Problem: For any X and $a, b \geq 0$, what is the relationship between $N_a(N_b(X))$ and $N_{a+b}(X)$?

(*Hint*: Triangle inequality!)

Can you now rephrase our conjecture more simply?

Conjecture (version 2):

Let's "prove" our conjecture. Our "proof" will tell us what the precise conjecture should have been.

Key Observation: If $A \subset B$, then $N_R(A) \subset N_R(B)$.

Idea: Pick a point p in X, and for any $t \geq 0$ let D_t denote the disk with center p and radius t.

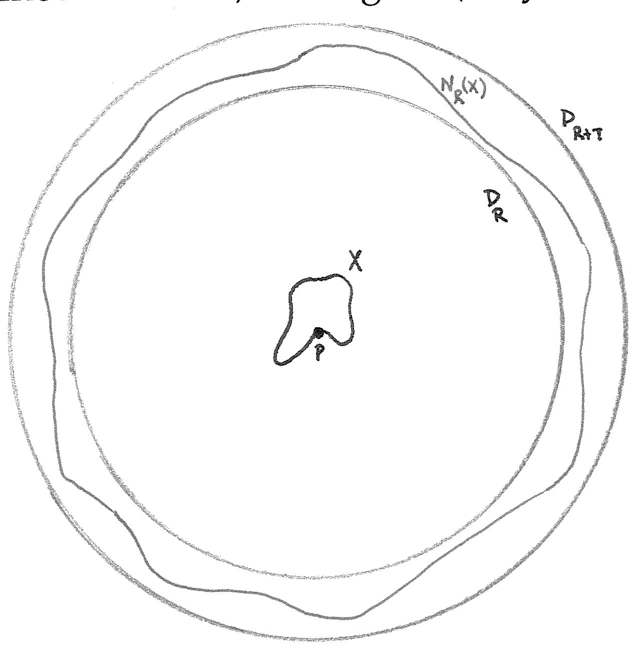
Choose some T such that $X \subset D_T$.

Using our key observation, what can you deduce about $N_R(X)$ from the inclusions

$$\{p\} \subset X \subset D_T?$$

What happens as you let $R \to \infty$?

Here's a zoomed out view, for large R (only boundaries shown):



Part II: Every Doodle is a Winding Road

Problem: Suppose X is a convex polygon. Can you express the perimeter of $N_r(X)$ in terms of X and r? How about the area?

(Try an example or two!)

$\text{Perim}(N_r(X)) =$

$\text{Area}(N_r(X)) =$

Observations:

1) $\frac{d}{dr}\mathsf{Area}(N_r(X)) = \mathsf{Perim}(N_r(X))$. Why is this?

   This is also true for circles. It's even true for squares, if one defines "radius" to be the distance from the center of the square to the midpoint of a side. Can you define such a "radius" for an equilateral triangle (so that the derivative of area is perimeter)? How about other regular polygons?

2) The formula for $\mathsf{Area}(N_r(X))$ is a quadratic polynomial, and the coefficients are all geometrically interesting!

Problem: Suppose now X is not a polygon (but is still convex). What can you say about the area and perimeter of $N_r(X)$ now? (And can you prove it?)

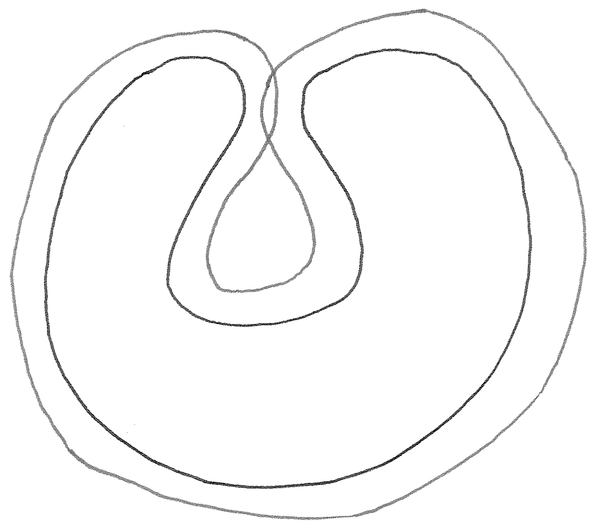
$\mathsf{Perim}(N_r(X)) =$

$\mathsf{Area}(N_r(X)) =$

String-Around-the-Earth Problem

Suppose a string is wrapped tightly around the equator of the earth. Someone with too much time on their hands adds 1 meter more of string, and raises the string above the earth a constant height. How high off the ground is the string?

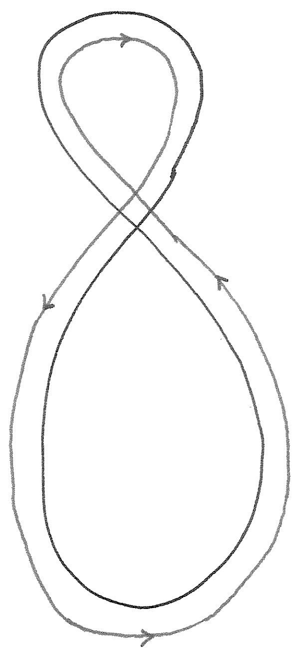
Problem: Suppose now we drop convexity, as in the figure below:



Our perimeter formula still holds, but the area formula is no longer correct.

Can you find a way to assign weights to different regions so that our original area formula still holds?

Problem: Now let's doodle around a figure that crosses itself, like this figure eight:



We get a different (but still nice) formula for the perimeter:

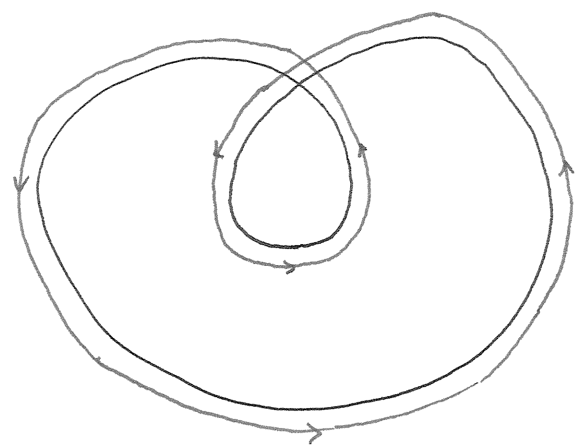
$\mathsf{Perim}(N_r(X)) =$

If we attach particular multiplicities to certain regions, we also get a nice area formula:

$\mathsf{Area}(N_r(X)) =$

And we still have $\frac{d}{dr}\mathsf{Area}(N_r(X)) = \mathsf{Perim}(N_r(X))$!

Problem: Can you find the perimeter and area formulas for this double loop (weighting regions appropriately)?



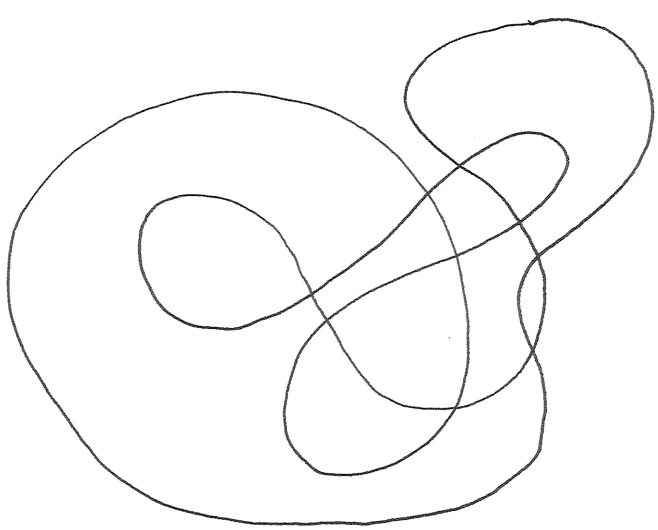
Perim$(N_r(X)) =$

Area$(N_r(X)) =$

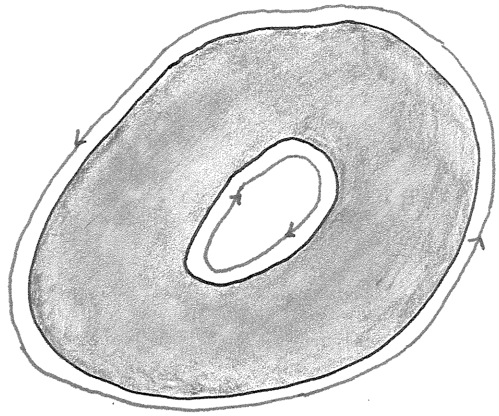
What rule tells you how to weight each region?

(You just discovered the winding number!)

Can you now immediately guess formulas (and weights) for crazy doodles?



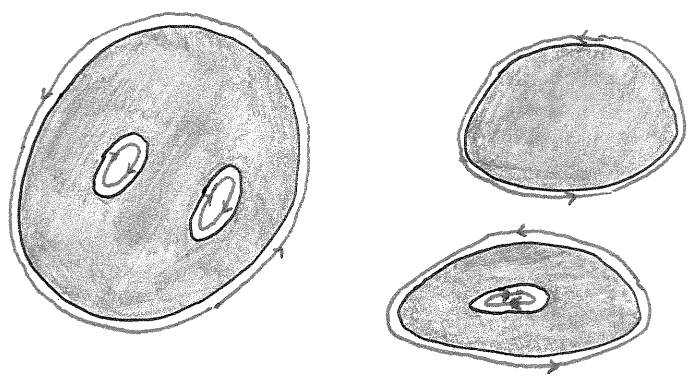
Problem: Can you find perimeter and area formulas for a region with a hole in it?



$\text{Perim}(N_r(X)) =$

$\text{Area}(N_r(X)) =$

How about for this archipelago?



$\text{Perim}(N_r(X)) =$

$\text{Area}(N_r(X)) =$

(You just discovered the Euler characteristic!)

Part III: Another Dimension

Problem: Suppose X is a box. Can you find formulas for the surface area and volume of $N_r(X)$?

$\mathsf{Surf}(N_r(X)) =$

$\mathsf{Vol}(N_r(X)) =$

Observations:

1) We again have $\frac{d}{dr}\mathsf{Vol}(N_r(x)) = \mathsf{Surf}(N_r(X))$!

2) The volume of $N_r(X)$ is a cubic polynomial in r, with some interesting coefficients. But what's up with the coefficient of the quadratic term? We have a new mystery invariant!

Russian Train Problem

A Russian train company has a rule: you are not allowed packages (in a box) whose sum of dimensions (length plus width plus height) exceeds 1 meter. Can you cheat by taking an illegal box and putting it in a larger box that is legal?

(*Hint*: If $X \subset Y$ are boxes, try comparing $\mathsf{Vol}(N_r(X))$ with $\mathsf{Vol}(N_r(Y))$.)

Part IV: Unlearn What You Have Learned

For convex X in n dimensions, you might (correctly) guess that

$$\text{HyperVol}(N_r(X)) = a_0 + a_1 r + a_2 r^2 + \cdots a_n r^n.$$

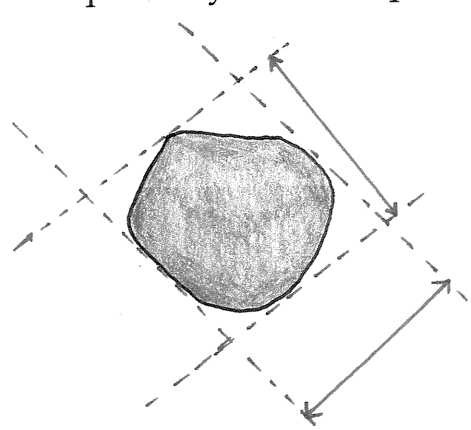
You might also (correctly) guess that a_0 is the hypervolume of X, a_1 is the hypersurface area of X, and a_n is the volume of the unit n-ball. But what are the meanings of the other coefficients?

One answer comes from the field of geometric probability (in a beautiful lecture by G.-C. Rota).

Dimension 2

Amazing Fact: The average length of the shadow of a convex (planar) body, multiplied by π, is the perimeter!

Example:



Questions:

1) What is the average length of the shadow of a circle of radius r?

2) Using the above fact (or calculus), what is the average length of the shadow of a line segment of length 1?

Buffon's Needle Problem (1733)

Suppose the floor is marked with parallel lines spaced 1 inch apart, and a 1-inch long needle is randomly dropped on the floor. What is the probability it crosses one of the lines?

This gives an experimental method for measuring π!

Dimension 3

Amazing Fact: The average (planar) shadow of a convex (space) body, multiplied by 4, is the surface area!

Try this for a sphere. (If that's too easy, try it for a box.)

Problem: Our "mystery invariant" for a convex body in three dimensions equals the average length of its shadow on a one-dimensional screen, multiplied by a certain constant. What is this constant?

(*Hint:* Try working it out for a sphere.)

Problem: Give an interpretation of the k-dimensional invariant in dimension n.