The Mathematics of Doodling

Mills College
The Möbius Band

November 12, 2010

Doodle Warm-up

Draw a random figure below, and then trace a curve around it, as closely as you can. Then trace another curve. And another. Repeat a bunch of times.

What seems to be happening to the traced curves?

Definition: Given a set X and number $r \geq 0$, define the *distance r neighborhood* of X:

$$N_r(X) = \{y \mid \text{dist}(y, x) \leq r \text{ for some } x \in X\}.$$

Question: In some sense, does the boundary of

$$N_r(N_r(\cdots(N_r(X))\cdots))$$

become circular as we iterate N_r?

This is the end of our warm-up. If you'd like to learn the answer to this question, follow the narwhal to Bonus Project #1!

Every Doodle is a Winding Road

Problem: Suppose X is a convex polygon. Can you express the perimeter of $N_r(X)$ in terms of X and r? How about the area?

(Try an example or two!)

Perim$(N_r(X)) =$

Area$(N_r(X)) =$

Observations:

1. The formula for $\mathsf{Area}(N_r(X))$ is a

2. $\frac{d}{dr}\mathsf{Area}(N_r(X)) =$

If you think this equation is cool, check out Bonus Project #2!

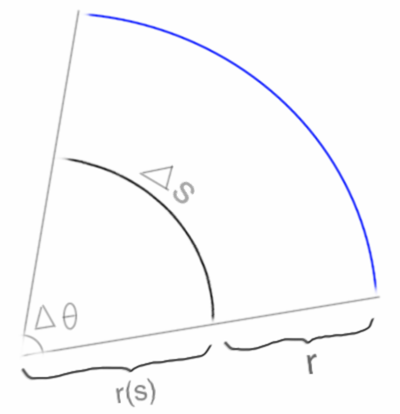
Problem: Suppose now X is not a polygon but is still convex. What can you say about the area and perimeter of $N_r(X)$ now? (And can you prove it?)

$\mathsf{Perim}(N_r(X)) =$

$\mathsf{Area}(N_r(X)) =$

Secret Calculus Aside:

Suppose we parameterize X by arc length s. Near any point s of X, the curve X "looks like" a circle of some radius $r(s)$. The number $\kappa(s) := \frac{1}{r(s)}$ is called the curvature of X at s. So a close-up view of X and its doodle $N_r(X)$ is:



This little piece of the picture contributes a difference in perimeters of approximately

$$(r(s) + r)\Delta\theta - r(s)\Delta\theta = r\Delta\theta = r\kappa(s)\Delta s.$$

Integrating, we find that

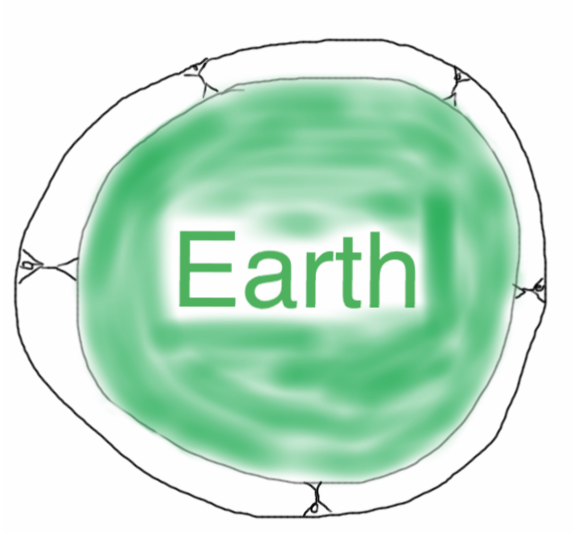
$$\text{Perim}(N_r(X)) - \text{Perim}(X) = r \int \kappa(s) ds.$$

This final integral is the total curvature of X. It is always 2π times an integer T, called the turning number of the curve. Can you guess why it is called the turning number?

Try working out the similar calculation for area!

Circling the Earth

An eccentric billionaire decides to wrap a string tightly around the equator of the earth. Her mischievous billionaire friend then comes along, adds an additional meter of string, and commands her robot army to uniformly raise the string a constant height above the surface.

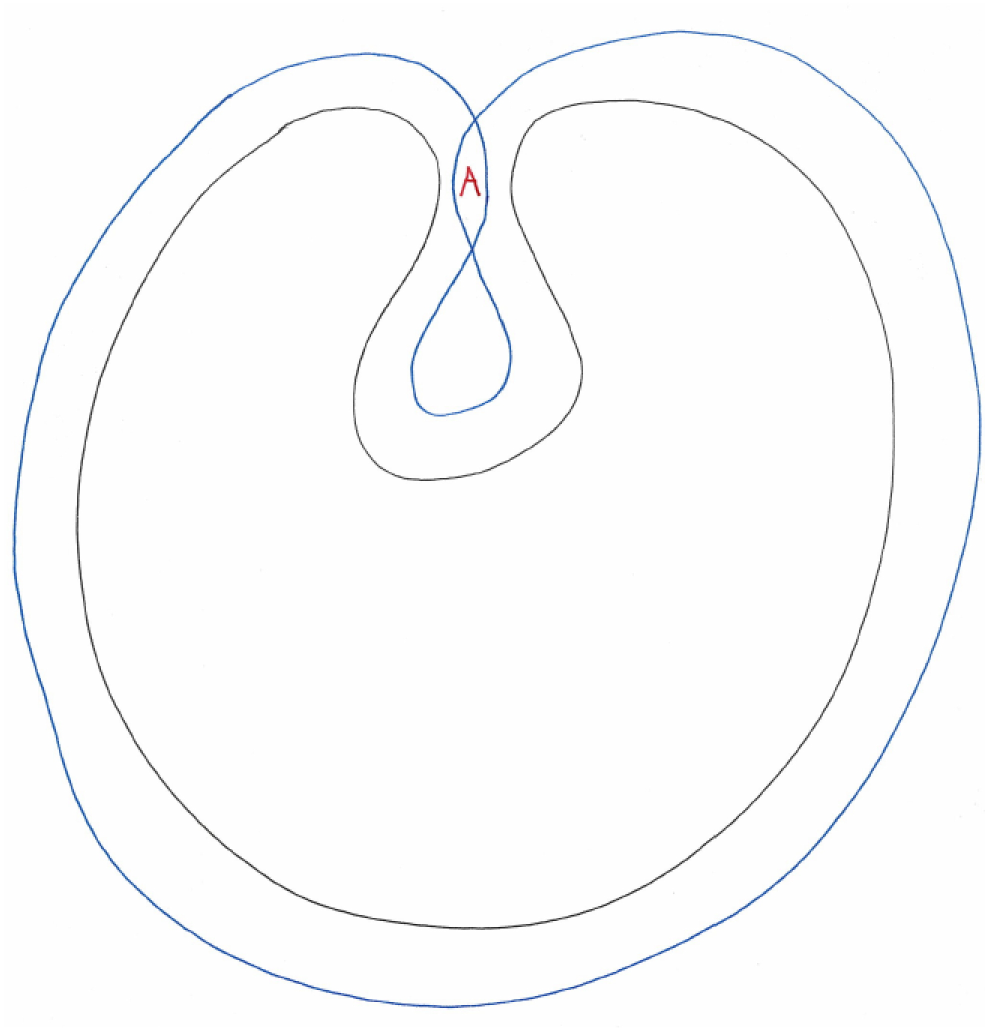


Question: How high off the ground is the string?

(Hint: You don't need to know the radius of the earth to answer this question. In fact, you don't even need to approximate the earth with a sphere!)

Answer:

Problem: Suppose now we drop convexity.

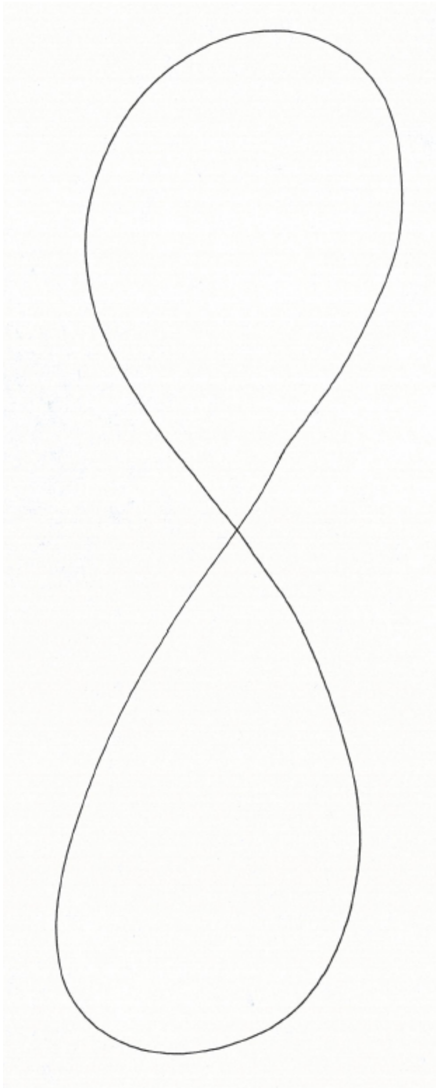


Perim$(N_r(X)) =$

If we weight region A by ____ , then

Area$(N_r(X)) =$

Problem: Now let's doodle around a figure that crosses itself, like this figure eight:

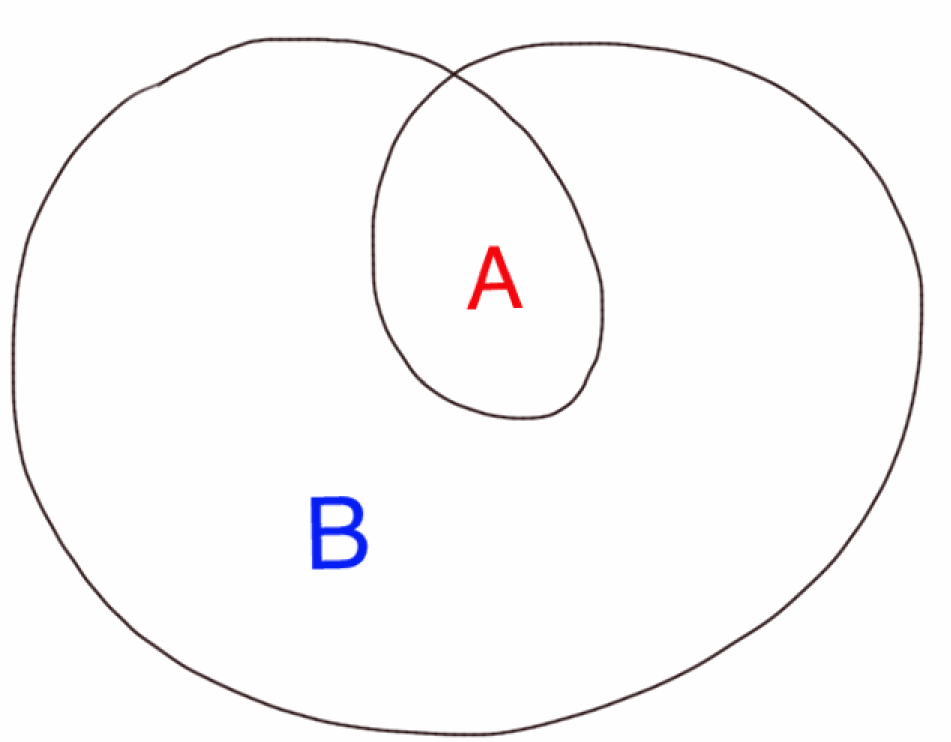


$\text{Perim}(N_r(X)) =$

If we weight region A by ____ and region B by ____ , then

$\text{Area}(N_r(X)) =$

Problem: Can you find the perimeter and area formulas for this double loop?



$\text{Perim}(N_r(X)) =$

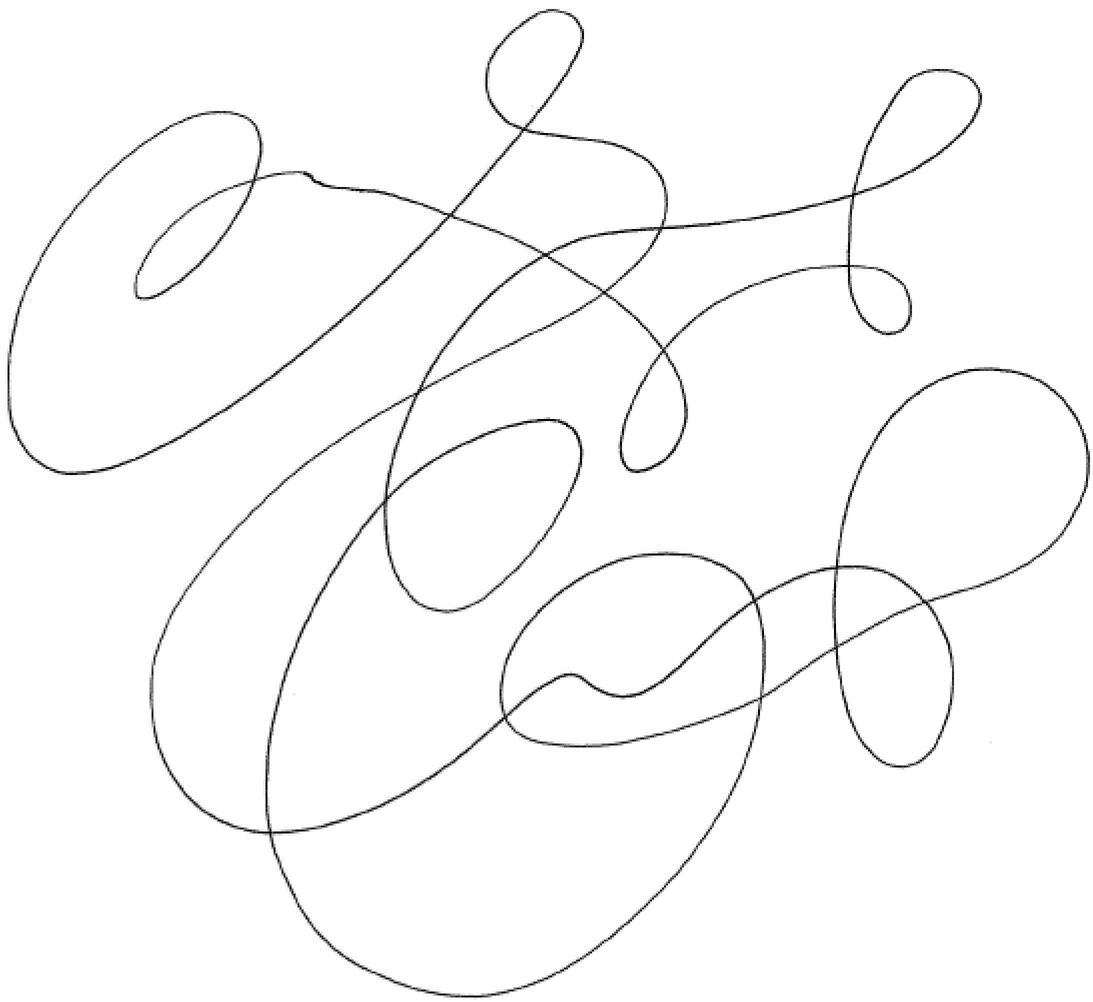
If we weight region A by ____ and region B by ____ , then

$\text{Area}(N_r(X)) =$

Question: What rule tells you how to weight each region?

Answer:

Can you now immediately guess formulas (and weights) for crazy doodles?



Like geography? See Bonus Project #3!

To Another Dimension

Problem: Suppose X is a solid box. Can you find formulas for the surface area and volume of $N_r(X)$?

$\mathsf{Surf}(N_r(X)) =$

$\mathsf{Vol}(N_r(X)) =$

Observations:

1. $\frac{d}{dr}\mathsf{Vol}(N_r(x)) =$

2. The volume of $N_r(X)$ is a

Question: What's up with the coefficient of the quadratic term?

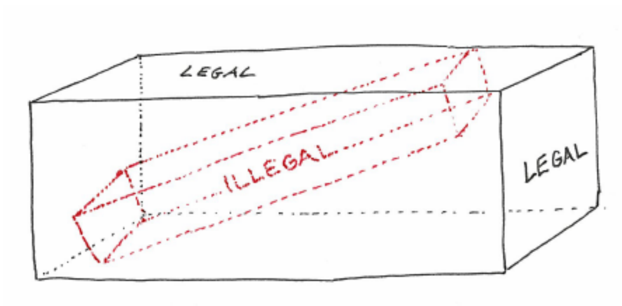
Those Rascally Russians

You need to get from Moscow to Perm, and your only option is the Trans-Siberian Railroad. You quickly learn that the Russian train tycoons have two very specific rules when it comes to luggage:

(1) All luggage must be box-shaped.

(2) You are not allowed boxes whose sum of dimensions (length plus width plus height) exceeds one meter.

Question: Can you cheat by taking an illegal box and jamming it in a larger, legal box?

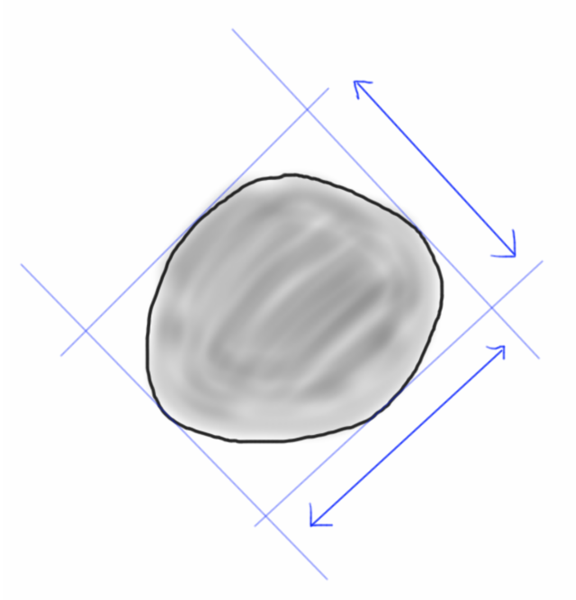


Answer:

Prepare for Insanity

Amazing Fact #1: The average length of the shadow of a convex planar body, multiplied by π, is the perimeter!

Example:



Questions:

1) What is the average length of the shadow of a circle of radius r?

2) Using the above fact (or calculus), what is the average length of the shadow of a line segment of length 1?

Buffon's Needle

It's the year 1733, and you've been imprisoned for your radical political views. One day you notice the floor of your cell is lined with parallel lines, spaced one inch apart. As chance would have it, your sole possession happens to be an inch-long needle. For enjoyment, then, you spend your prison time ("slow time") dropping the needle onto the floor and keeping a tally of how many times it crosses a line.

Question: If you drop the needle a thousand times, how many of those times would you expect it to cross a line?

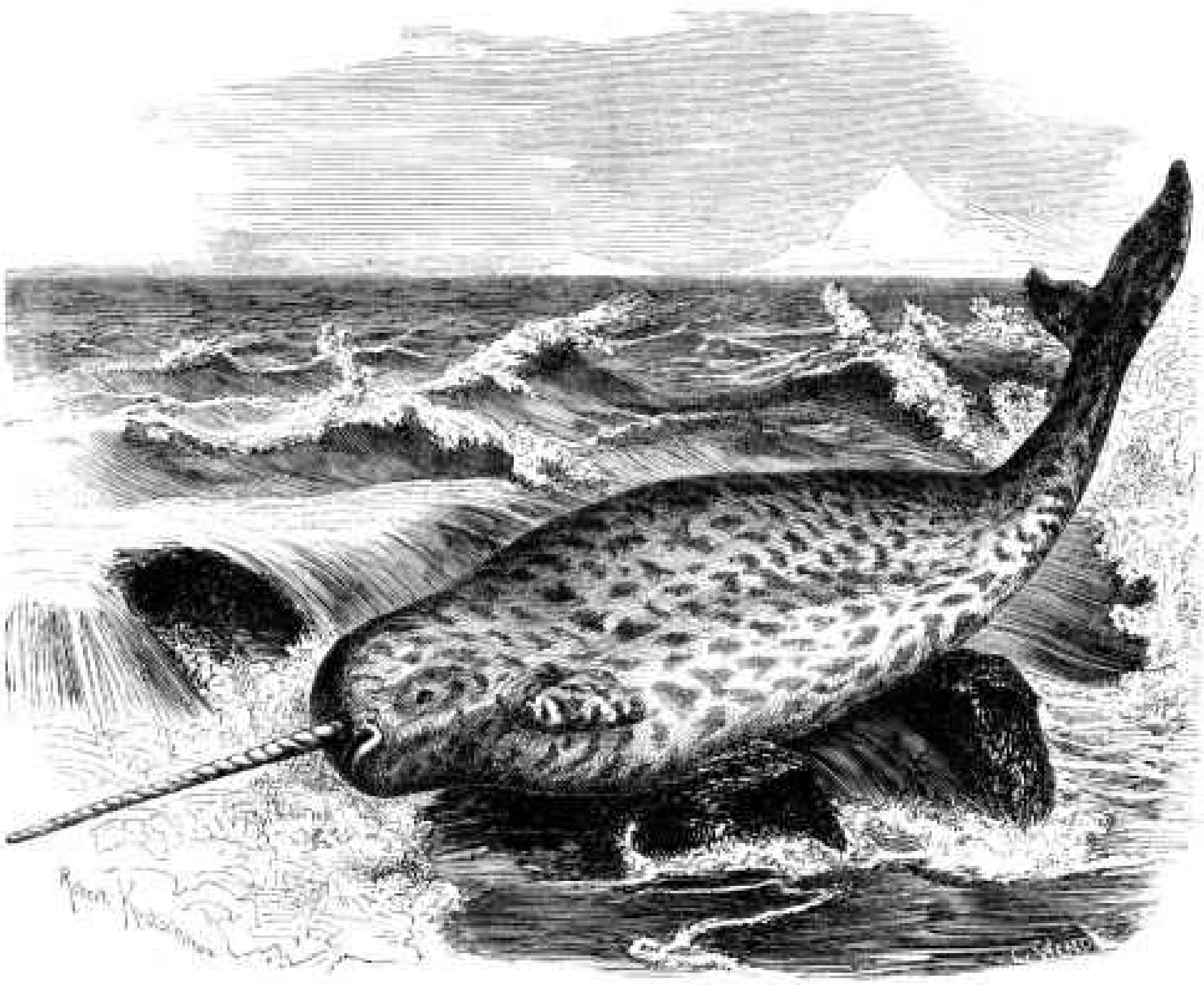
This gives an experimental method of measuring π!

Amazing Fact #2: The average planar shadow of a convex space body, multiplied by 4, is the surface area!

Try this for a sphere. (If that's too easy, try it for a box.)

Problem: Our "mystery invariant" for a convex body in three dimensions equals the average length of its shadow on a one-dimensional screen, multiplied by a certain constant. What is this constant?

Bonus Projects



Bonus Project 1 - Circular Reasoning

Suppose X is a bounded subset of the plane. Based on our warm-up doodling, we might:

Conjecture: In some sense, the boundary of

$$N_r(N_r(\cdots(N_r(X))\cdots))$$

become circular as we iterate N_r.

Question: For any $a, b \geq 0$, what is the relationship between $N_a(N_b(X))$ and $N_{a+b}(X)$?

(*Hint*: Triangle inequality!)

Answer:

Can you now rephrase our conjecture more simply?

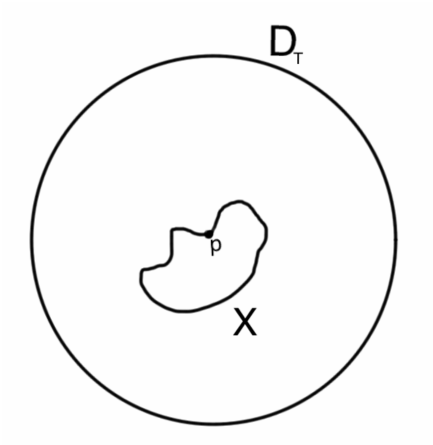
Conjecture:

Let's "prove" our conjecture. Our "proof" will tell us what the precise conjecture should have been.

Key Observation: If $A \subseteq B$, then $N_r(A) \subseteq N_r(B)$.

Idea: Pick a point p in X, and for any $t \geq 0$ let D_t denote the disk with center p and radius t.

Choose some T such that $X \subset D_T$.

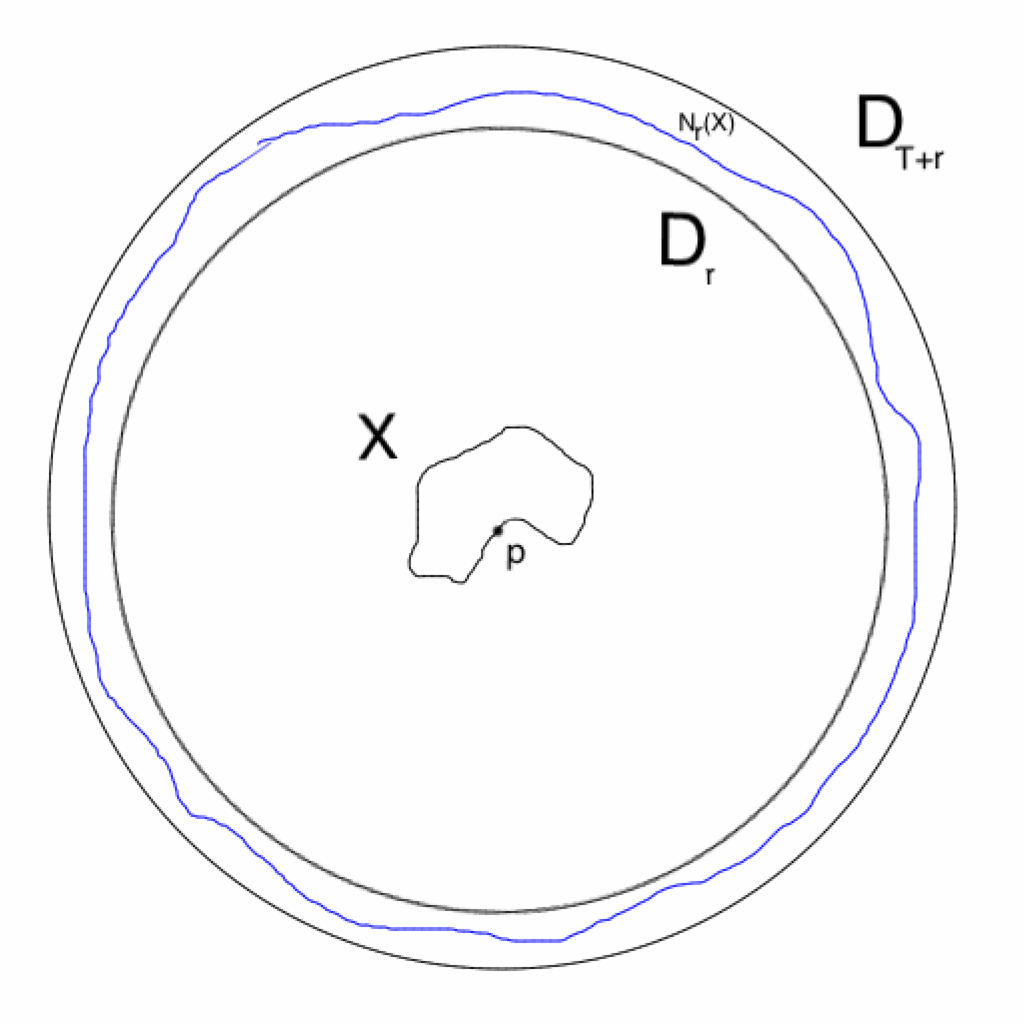


Using our key observation, what can you deduce about $N_r(X)$ from the inclusions

$$\{p\} \subset X \subset D_T?$$

What happens as you let $r \to \infty$?

Here's a zoomed out view, for large r:



This is the sense in which $N_r(X)$ becomes circular as $r \to \infty$.

Bonus Project 2 - Radii-cal Thinking

Earlier we found that if X is a convex polygon, then

$$\frac{d}{dr}\mathsf{Area}(N_r(X)) = \mathsf{Perm}(N_r(X)).$$

You'd probably already noticed this relationship between the area and circumference of a disk.

Problem: Find a way to define the "radius" of a square, so that the derivative of the area equals the perimeter.

Problem: Do the same for an equilateral triangle.

Challenge: Do the same for an arbitrary regular polygon.

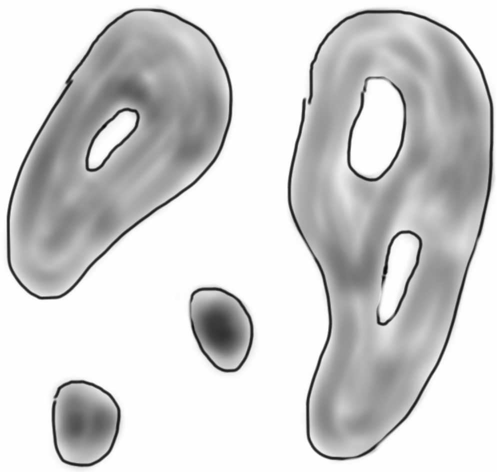
Bonus Project 3 - Euler Spills

Problem: Can you find perimeter and area formulas for a region with a hole in it?

Perim$(N_r(X)) =$

Area$(N_r(X)) =$

How about for this archipelago?



Perim$(N_r(X)) =$

Area$(N_r(X)) =$

(You just discovered the Euler characteristic!)