

RECASTING RESULTS IN EQUIVARIANT GEOMETRY

AFFINE COSETS, OBSERVABLE SUBGROUPS AND EXISTENCE OF GOOD QUOTIENTS

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Abstract. Using the language of stacks, we recast and generalize a selection of results in equivariant geometry.

1. Introduction

When an algebraic group G acts on a variety X, there is a precise dictionary between the G-equivariant geometry of X and the geometry of the quotient stack $[X/G]$. This is typical of the strong interplay between equivariant geometry and algebraic stacks. Indeed, results (as well as their proofs) in the theory of algebraic stacks are often inspired by analogous results in equivariant geometry. As the simplest stacks are quotient stacks, they are fertile testing grounds for more general results. Conversely, algebraic stacks can be quite useful for proving results in equivariant geometry. The purpose of the present paper is to provide some examples of this power, reproving and generalizing several theorems in equivariant geometry via the language of algebraic stacks.

After giving a brief overview of algebraic stacks in Section 2, we begin in Section 3 by summarizing the relationship between the equivariant geometry of a scheme and the geometry of its corresponding quotient stack. In Section 4, we review the classical notion of a good quotient and the more modern notion of a good moduli space, and explore the relationship between them. As a result, we recover and generalize [BBŚ, Theorem B]:

Theorem 1.1. *Let $G \to S$ be an affine, linearly reductive group scheme acting on an algebraic space X, and suppose X admits a G-invariant affine morphism $f : X \to Z$ to an algebraic space. Then there exists a good quotient $\pi : X \to Y$ with Y an algebraic space.*

Note in particular that the normality hypothesis of [BBŚ, Theorem B] has been

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removed, thereby answering [BBŚ, Question, p. 149].

In Section 5, we quickly recover Matsushima's theorem (see [Mat60], [BB63], [Hab78] and [Ric77]) using tools developed in Section 4:

Theorem 1.2. *Suppose $G \to S$ is an affine, linearly reductive group scheme and $H \subseteq G$ is a flat, finitely presented, separated subgroup scheme. Then the following are equivalent:*

(i) *$H \to S$ is linearly reductive;*
(ii) *$G/H \to S$ is affine;*
(iii) *the functor $\mathcal{F} \mapsto \operatorname{Ind}_H^G \mathcal{F}$ from $\operatorname{QCoh}^H(S)$ to $\operatorname{QCoh}^G(S)$ is exact.*

This is a prototypical example of the power of algebraic stacks in the study of equivariant problems. (This theorem was also proved in [Alp08, Theorem 12.15] using the same techniques, but we include the proof again here to emphasize the convenience of the language of stacks.)

Section 6 focuses on properties of observable subgroup schemes. When working over a field, a subgroup scheme $H \subseteq G$ is *observable* if every finite dimensional H-representation is a sub-H-representation of a finite dimensional G-representation. We extend the definition to an arbitrary base scheme in Definition 6.4. We find the following characterization of such subschemes:

Theorem 1.3. *Let $G \to S$ be a flat, finitely presented, quasi-affine group scheme and $H \subseteq G$ a flat, finitely presented, quasi-affine subgroup scheme. The following are equivalent:*

(i) *H is observable;*
(ii) *for every quasi-coherent $\mathcal{O}_S[H]$-module $\mathcal{F}$, the counit morphism of the adjunction, $\operatorname{Ind}_H^G \mathcal{F} \to \mathcal{F}$, is a surjection of $\mathcal{O}_S[H]$-modules;*
(iii) *$BH \to BG$ is quasi-affine;*
(iv) *$G/H \to S$ is quasi-affine.*

If S is noetherian, then the above are also equivalent to:

(v) *every coherent $\mathcal{O}_S[H]$-module is a quotient of a coherent $\mathcal{O}_S[G]$-module; and*
(vi) *for every coherent $\mathcal{O}_S[H]$-module $\mathcal{F}$, the counit morphism of the adjunction, $\operatorname{Ind}_H^G \mathcal{F} \to \mathcal{F}$, is a surjection of $\mathcal{O}_S[H]$-modules.*

The proof of the above theorem follows directly from the observation that a representable morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is quasi-affine if and only if the adjunction morphism $f^* f_* \mathcal{F} \to \mathcal{F}$ is surjective for all quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{F}$ (see Proposition 6.2).

Lastly, in Section 7 we analyze the existence of good moduli spaces, ultimately recovering a modified version of [BBŚ, Theorem C]:

Theorem 1.4. *Let G be a connected algebraic group acting on a scheme X of finite type over an algebraically closed field k, and suppose that for every pair of points $x, y \in X$ there exists a G-invariant open subscheme $U_{xy} \subseteq X$ that contains x and y and admits a good quotient. Then X admits a good quotient.*

Note that here we assume the group G is connected, but not necessarily reductive. In fact, it appears that the proof of [BBŚ, Theorem C] is incomplete, as the constructibility of certain subsets is never verified (see Remark 7.14). It is in the verification of that constructibility that we need to impose the connectedness hypothesis. This is also the reason we need to work with group actions rather than more general algebraic stacks. We expect, however, a stronger version of the above theorem (as well as of Lemma 7.13) to hold.

Remark 1.5. It is also possible to show that an analogue of [BBŚ, Theorem A] holds using similar—but significantly more involved—methods.

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2. Algebraic stacks

We begin with a few words on algebraic stacks. The reader familiar with these objects may comfortably skip to Section 3, below. For those less familiar, we offer the following brief overview. Algebraic stacks are, at their most basic level, a generalization of schemes. Algebraic stacks arise naturally in the context of moduli problems as they provide geometric objects which represent the moduli problem.

In a moduli problem, one is generally interested in analyzing a collection of objects of a certain type, e.g., of curves of a fixed genus g. One would like this collection to be endowed with a structure beyond that of merely a set, but rather something closer to that of a space, so that, for example, a curve in the space would correspond to a "nicely varying" (i.e., flat) family of objects over the curve. In algebraic geometry, we would ideally prefer moduli spaces to have the structure of a variety, so that we may then bring to bear on them our full arsenal of algebrogeometric tools. Properties deduced about moduli spaces are clearly of great significance, as they reveal universal facts about the objects being parameterized.

Unfortunately, it was discovered early on that some moduli spaces could not exist as schemes, e.g., the moduli space of elliptic curves. The culprit is the existence of automorphisms of some of the objects being parameterized; this prevents the representability of the moduli functor by a scheme. In the seminal papers by Deligne and Mumford [DM69] and Artin [Art74], the concept of an *algebraic stack* was introduced. Algebraic stacks have proved to be exceptionally convenient geometric objects for studying moduli problems.

An algebraic stack is a categorical object possessing many of the excellent functorial properties one would hope for (and need) when studying moduli problems. Its definition also includes sufficient additional properties to tie it closely to the world of schemes (and algebraic spaces) and make the extension of algebraic geometry to stacks possible. Most notably, by definition every algebraic stack $\mathcal{X}$ admits a smooth surjection from an algebraic space, $X \to \mathcal{X}$, which one thinks of as an atlas for the stack, much as in the world of manifolds. Many properties of algebraic stacks are understood (or even defined) by pulling back to the covering algebraic space X.

The precise definition of algebraic stacks requires a fairly substantial amount of definitions and exposition. We direct the interested reader to the following references, which explain stacks in greater detail and clarity than we could hope to achieve here. This list is by no means exhaustive and should be viewed merely as a starting point in the study of stacks. A short and friendly introduction to the theory is [Fan01]. For a more technical yet still concise overview, the reader might consider [Vis89, Appendix]. A very thorough development is [LMB00]; as of this writing, it is the only currently published book on stacks. The most exhaustive reference, and likely the best source for the most up-to-date definitions and properties, is [spa].

Notation and Conventions 2.1. Throughout this paper, all schemes are assumed quasi-separated. Let S be a scheme. Recall that an *algebraic space* over S is a sheaf of sets X on the big étale site $\mathrm{S_{Et}}$ of schemes over S such that:

(i) $\Delta_{X/S}\colon X \to X \times_S X$ is representable by schemes and is quasi-compact; and
(ii) there exists an étale, surjective map $U \to X$ with U a scheme.

An *algebraic stack* over S is a stack $\mathcal{X}$ over $\mathrm{S_{Et}}$ such that:

(i) $\Delta_{\mathcal{X}/S} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, separated and quasi-compact; and
(ii) there exists a smooth, surjective map $X \to \mathcal{X}$ with X an algebraic space.

We note that there is confusion in the literature on the definition of an algebraic stack. Deligne–Mumford require in condition (ii) of an algebraic stack the existence of an étale cover; such stacks are now commonly referred to as *Deligne–Mumford stacks*. The definition of an algebraic stack above is the same as is given in [LMB00]. In [Art74], Artin introduces the above definition but assumes in addition that $\mathcal{X}$ is of locally finite type over an excellent Dedekind domain. Often in research articles authors will drop the separated and quasi-compact condition from the diagonal in condition (i). See [spa] for a development of the theory of algebraic stacks without any condition on the diagonal.

Given an algebraic stack $\mathcal{X}$, we denote by $|\mathcal{X}|$ the topological space whose points correspond to equivalence classes of $\mathcal{O}_S$-field-valued points of $\mathcal{X}$ (see [LMB00, Chap. 5]). Any point $x \in |\mathcal{X}|$ has a residue field $k(x)$, which is the coarse moduli space of the residue gerbe $\mathcal{G}_x$ in $\mathcal{X}$; furthermore, there exists a representative $\operatorname{Spec} k \to \mathcal{X}$ of x with $k/k(x)$ finite (see [LMB00, Chap. 11], [Ryd10a, Theorem B.2]).

3. G-equivariant geometry of X vs. geometry of $[X/G]$

For simplicity, assume momentarily $S = \operatorname{Spec} k$, with k an algebraically closed field. If $G \to \operatorname{Spec} k$ is a finite type, affine group scheme acting on a finite type k-scheme X, then we have the following dictionary between the G-equivariant geometry of X and the geometry of $[X/G]$:

G-equivariant geometry of X	**Geometry of $[X/G]$**
orbit of $x \in X(k)$	point $\operatorname{Spec} k \to [X/G]$
G-invariant morphism $X \to Z$	morphism $[X/G] \to Z$
$\Gamma(X, \mathcal{O}_X)^G$	$\Gamma([X/G], \mathcal{O}_{[X/G]})$
quasi-coherent $\mathcal{O}_X[G]$-module F	quasi-coherent $\mathcal{O}_{[X/G]}$-module $\mathcal{F}$
$\Gamma(X, F)^G$	$\Gamma([X/G], \mathcal{F})$
G-linearization on X	line bundle on $[X/G]$
geometric quotient $X \to Y$	coarse moduli space $[X/G] \to Y$
good quotient $X \to Y$	good moduli space $[X/G] \to Y$

Example 3.1. Suppose $X = \text{Spec } k$ is a point, and write $BG = [\text{Spec } k/G]$. Then quasi-coherent (resp., coherent) $\mathcal{O}_{BG}$-modules correspond to G-representations (resp., of finite dimension).

Remark 3.2.

(i) Technically, geometric quotients and coarse moduli spaces do not precisely agree under this dictionary. For an action of G on X, a geometric quotient $X \to Y$ is not necessarily a categorical quotient (see [Kol97, Example 2.18]), which is a prerequisite condition for $[X/G] \to Y$ to be a coarse moduli space. However, they do agree in the case of proper group actions; see [Kol97, Remark 2.20], [Ryd11, §7].

(ii) The dictionary between good quotients and good moduli spaces is made precise in Section 4.

Returning now to the general case, suppose $G \to S$ is a flat, separated and quasi-compact group scheme over an arbitrary base S. As usual, denote $BG = [S/G]$. Then a quasi-coherent (resp., coherent) $\mathcal{O}_{BG}$-module corresponds to a quasi-coherent (resp., coherent) $\mathcal{O}_S[G]$-module. (Recall that an $\mathcal{O}_S[G]$-module is an $\mathcal{O}_S$-module $\mathcal{F}$ together with a lifting of the action of G on S to an action of G on $\mathcal{F}$; see [Mum65, §1.3] for the case of invertible sheaves.)

A morphism $H \to G$ of flat, separated and quasi-compact group schemes induces a morphism $f : BH \to BG$ of algebraic stacks. If, in addition, $H \hookrightarrow G$ is a subgroup scheme, then the diagram below is cartesian:

$$\begin{array}{ccc} G/H & \longrightarrow & S \\ \downarrow & & \downarrow \\ BH & \xrightarrow{f} & BG. \end{array}$$

Using descent theory, we can therefore relate properties of the morphism $f : BH \to BG$ to properties of the quotient $G/H \to S$.

If $\mathcal{G}$ is an $\mathcal{O}_S[G]$-module, then $f^*\mathcal{G}$ is the $\mathcal{O}_S[H]$-module with the same underlying $\mathcal{O}_S$-module as $\mathcal{G}$, but with H-action induced from the morphism $H \to G$. If $\mathcal{F}$ is an $\mathcal{O}_S[H]$-module, then $\text{Ind}_H^G \mathcal{F} := f_*\mathcal{F}$ is the induced $\mathcal{O}_S[G]$-module. There is a natural morphism $\text{Ind}_H^G \mathcal{F} \to \mathcal{F}$, corresponding to the adjunction morphism $f^*f_*\mathcal{F} \to \mathcal{F}$.

4. Good quotients vs. good moduli spaces

Good quotients

Although the notion of a *good quotient* was first explicitly written down by Seshadri in [Ses63, Definition 1.5], the idea was already implicit in Mumford's theory of quotients by reductive groups (see [Mum65]).

Definition 4.1. Suppose $G \to S$ is an affine group scheme acting on an algebraic space X. We say a morphism $\pi : X \to Y$ to an algebraic space is a *good quotient* if the following hold:

(i) π is surjective, affine and G-invariant;
(ii) $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G$ is an isomorphism; and
(iii) for every morphism $Y' \to Y$ with base change $\pi' : X \times_Y Y' \to Y'$, the following hold:
  (a) for each closed G-invariant subspace $Z' \subseteq X \times_Y Y'$, the image $\pi'(Z') \subseteq Y'$ is closed; and
  (b) for each pair of closed G-invariant disjoint subspaces $Z'_1, Z'_2 \subseteq X \times_Y Y'$, the images $\pi'(Z'_1), \pi'(Z'_2)$ are disjoint.

Remark 4.2.
(i) In the case $G \to S$ is linearly reductive, for $\pi : X \to Y$ to be a good quotient it suffices to require only that π is affine and G-invariant, and $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G$ is an isomorphism (see Corollary 4.14).

(ii) As in other papers (see [BBŚ], [BB02] and [spa, Tag 04AC]), we have removed the assumption in [Ses63, Definition 1.5] that X and Y are schemes. We note that the argument of [Mum65, Remark 0.5] implies only that a good quotient is universal for G-equivariant maps to *schemes*. However, [Alp08, Theorem 6.6] implies that good quotients are universal for G-equivariant maps to algebraic spaces when X is noetherian; see Corollary 4.15.

(iii) As in [BB02] and [spa, Tag 04AC], we require that property (iii) holds for arbitrary base change. We note, however, that it is equivalent to restrict to flat morphisms $Y' \to Y$.

Linearly reductive groups

Let QCoh(S) denote the category of quasi-coherent $\mathcal{O}_S$-modules, and $\mathrm{QCoh}^G(S)$ denote the category of quasi-coherent $\mathcal{O}_S[G]$-modules.

Definition 4.3. A flat, finitely presented, affine group scheme $G \to S$ is *linearly reductive* if the functor $\mathcal{F} \mapsto \mathcal{F}^G$ from $\mathrm{QCoh}^G(S)$ to $\mathrm{QCoh}(S)$ is exact.

Remark 4.4. It is not essential to assume $G \to S$ is affine (cf. [Alp08, Chapter 12]). The assumption is only made here to simplify the discussion.

The following is well known (see, for instance, [Alp08, Prop. 12.6]):

Proposition 4.5. *Let $G \to \operatorname{Spec} k$ be a finite type, affine group scheme, with k a field. The following are equivalent:*
(i) *G is linearly reductive;*
(ii) *the functor $V \mapsto V^G$ from G-representations to vector spaces is exact;*
(iii) *the functor $V \mapsto V^G$ from finite-dimensional G-representations to vector spaces is exact;*
(iv) *every G-representation is completely reducible;*
(v) *every finite-dimensional G-representation is completely reducible; and*
(vi) *for every finite-dimensional G-representation V and nonzero $v \in V^G$, there exists $F \in (V^\vee)^G$ with $F(v) \neq 0$.*

Cohomologically affine morphisms

For an algebraic stack $\mathcal{X}$, let $\mathrm{QCoh}(\mathcal{X})$ denote the category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules.

Definition 4.6. ([Alp08, Definition 3.1]) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *cohomologically affine* if it is quasi-compact and the push-forward functor $\mathcal{F} \mapsto f_*\mathcal{F}$ from $\mathrm{QCoh}(\mathcal{X})$ to $\mathrm{QCoh}(\mathcal{Y})$ is exact.

Remark 4.7. Cohomological affineness possesses several nice properties, such as:

(i) (*Serre's criterion*) If $f : \mathcal{X} \to \mathcal{Y}$ is an arbitrary morphism of algebraic spaces, then f is cohomologically affine if and only if f is affine. More generally, if $f : \mathcal{X} \to \mathcal{Y}$ is a representable morphism of algebraic stacks and $\mathcal{Y}$ has quasi-affine diagonal, then f is cohomologically affine if and only if f is affine.

(ii) A flat, finitely presented, affine group scheme $G \to S$ is linearly reductive if and only if $BG \to S$ is cohomologically affine.

(iii) Cohomologically affine morphisms are stable under composition and are local on the target under faithfully flat morphisms.

(iv) If $f : \mathcal{X} \to \mathcal{Y}$ is a cohomologically affine morphism of algebraic stacks and $\mathcal{Y}$ has quasi-affine diagonal, then for any morphism $\mathcal{Y}' \to \mathcal{Y}$ of algebraic stacks, the base change $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ is cohomologically affine.

See [Alp08, Chap. 3] for references and additional details.

We note the following interesting and well-known consequence of these properties, which stresses the necessity of quasi-affineness of the diagonal in (iv), above:

Proposition 4.8. *If $G \to \mathrm{Spec}\ A$ is a finite type, quasi-affine group scheme over an Artin ring A, then $G \to \mathrm{Spec}\ A$ is affine.*

Proof. We may assume $A = k$ is a field, since a scheme is affine if and only if its reduction is affine. Observe that if $G \to \mathrm{Spec}\ k$ is any finite type group scheme, then $\pi : \mathrm{Spec}\ k \to BG$ is cohomologically affine. Indeed, the push-forward by π of a k-vector space V corresponds to the G-representation $V \otimes_k \Gamma(G, \mathcal{O}_G)$, where $\Gamma(G, \mathcal{O}_G)$ is the left regular representation of G, and this functor is clearly exact. If $G \to \mathrm{Spec}\ k$ is quasi-affine, then $BG \to \mathrm{Spec}\ k$ has quasi-affine diagonal. Thus the base change $G \cong \mathrm{Spec}\ k \times_{BG} \mathrm{Spec}\ k \to \mathrm{Spec}\ k$ is cohomologically affine, and hence by Serre's criterion (Remark 4.7) affine. $\square$

Good moduli spaces

Definition 4.9. ([Alp08, Definition 4.1]) Let $\mathcal{X}$ be an algebraic stack. We say a quasi-compact morphism to an algebraic space, $\phi : \mathcal{X} \to Y$, is a *good moduli space* if:

(i) ϕ is cohomologically affine; and
(ii) $\mathcal{O}_Y \to \phi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

We summarize the basic properties of good moduli spaces:

Proposition 4.10 ([Alp08, Theorems 4.16, 6.6], [Alp10a, Theorem 6.3.3])**.** *Let $\mathcal{X}$ be an algebraic stack and $\phi : \mathcal{X} \to Y$ a good moduli space. Then:*

(i) *ϕ is surjective, universally closed and universally submersive;*
(ii) *if Z_1, Z_2 are closed substacks of $\mathcal{X}$, then*

$$\mathrm{im}\ Z_1 \cap \mathrm{im}\ Z_2 = \mathrm{im}\ (Z_1 \cap Z_2),$$

where the intersections and images are scheme theoretic;

(iii) *for each algebraically closed $\mathcal{O}_S$-field k, there is an equivalence relation defined on the set of isomorphism classes of k-valued points $[\mathcal{X}(k)]$, given by $x_1 \sim x_2 \in [\mathcal{X}(k)]$ when $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \varnothing$ in $|\mathcal{X} \times_S k|$, which induces a bijective map $[\mathcal{X}(k)]/\sim \to Y(k)$;*

(iv) *if $\mathcal{X}$ is locally noetherian, then ϕ is universal for maps to algebraic spaces (that is, for any morphism to an algebraic space $\psi : \mathcal{X} \to Z$, there exists a unique map $\xi : Y \to Z$ such that $\xi \circ \phi = \psi$);*

(v) *if $\mathcal{X}$ is locally noetherian, then Y is locally noetherian and ϕ_* preserves coherence;*

(vi) *if $\mathcal{X}$ is finite type over a noetherian scheme S, then Y is finite type over S; and*

(vii) *if $Y' \to Y$ is any morphism of algebraic spaces, then the base change $\mathcal{X} \times_{Y'} Y \to Y'$ is a good moduli space.*

Remark 4.11. The property of being a good moduli space also descends under faithfully flat morphisms. See [Alp08, Chapter 4] for further properties and a systematic development of the theory. We emphasize that other than (iv) and (vi), the proofs of the above properties are quite elementary.

Relationship between good quotients and good moduli spaces

We begin with:

Lemma 4.12. *Let $G \to S$ be an affine, linearly reductive group scheme acting on an algebraic space X, and let $X \to Y$ be a G-invariant morphism. Then the corresponding morphism $[X/G] \to Y$ is cohomologically affine if and only if $X \to Y$ is affine.*

Proof. Suppose $[X/G] \to Y$ is cohomologically affine. Since $X \to [X/G]$ is affine (as $G \to S$ is affine), the composition $X \to [X/G] \to Y$ is cohomologically affine, and therefore affine (by Serre's criterion, Remark 4.7(i)). Conversely, if $X \to Y$ is affine, consider the trivial action of G on Y. Since $G \to S$ is linearly reductive, $BG \to S$ is cohomologically affine. Since $[X/G] \to Y$ is the composition of the affine morphism $[X/G] \to [Y/G]$ and the cohomologically affine morphism $[Y/G] \to Y$ (the base change of $BG \to S$ by $Y \to S$), it is cohomologically affine. □

Proposition 4.13. *Let $G \to S$ be an affine, linearly reductive group scheme acting on an algebraic space X. Let $\pi : X \to Y$ be a G-invariant morphism. Then $\phi : [X/G] \to Y$ is a good moduli space if and only if:*

(i) *π is affine; and*

(ii) *$\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G$ is an isomorphism.*

Proof. Since there is a canonical isomorphism $\phi_*\mathcal{O}_{[X/G]} \cong \pi_*(\mathcal{O}_X)^G$, condition (ii) above is equivalent to condition (ii) of Definition 4.9. The previous lemma shows that condition (i) is equivalent to condition (i) of Definition 4.9. □

In particular, we can reinterpret the definition of a good quotient in the case of an action by an affine, linearly reductive group scheme.

Corollary 4.14. *If $G \to S$ is an affine, linearly reductive group scheme acting on an algebraic space X, then $\pi : X \to Y$ is a good quotient if and only if:*

(i) *π is an affine G-invariant morphism; and*
(ii) *$\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G$ is an isomorphism.*

Proof. This follows from Propositions 4.10 and 4.13. □

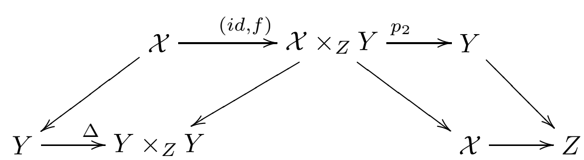
Of course, one can show the above corollary directly without recourse to the theory of stacks and good moduli spaces. However, we note that Proposition 4.10(iv) implies the uniqueness of good quotients in the category of algebraic spaces. We know of no direct proof of this result, so the language of stacks and good moduli spaces seems quite advantageous in this case.

Corollary 4.15. *Let $G \to S$ be an affine, linearly reductive group scheme acting on a noetherian algebraic space X. Suppose $\pi : X \to Y$ is a good quotient. Then for any G-invariant morphism $\varphi : X \to Z$ to an algebraic space Z, there is a unique morphism $\Psi : Y \to Z$ such that $\varphi = \Psi \circ \pi$.*

We can also immediately deduce:

Proposition 4.16. *Suppose an algebraic stack $\mathcal{X}$ admits a cohomologically affine morphism $f : \mathcal{X} \to Z$ to an algebraic space Z. Then there exists a good moduli space $\phi : \mathcal{X} \to Y$.*

Proof. Let $Y = \mathcal{S}pec_Z f_* \mathcal{O}_{\mathcal{X}}$ and $\phi : \mathcal{X} \to Y$ be the canonical morphism (hence $\mathcal{O}_Y \to \phi_* \mathcal{O}_{\mathcal{X}}$ is trivially an isomorphism). Consider the 2-cartesian diagram, in which the top composition is ϕ:



Since $Y \to Z$ is affine, so is $Y \to Y \times_Z Y$ and hence $\mathcal{X} \to \mathcal{X} \times_Z Y$. Since $\mathcal{X} \to Z$ is cohomologically affine and Z has quasi-affine diagonal (as it is an algebraic space), $\mathcal{X} \times_Z Y \to Y$ is cohomologically affine. Therefore, ϕ is cohomologically affine. □

We can now prove Theorem 1.1:

Theorem 1.1. *Let $G \to S$ be an affine, linearly reductive group scheme acting on an algebraic space X, and suppose X admits a G-invariant affine morphism $f : X \to Z$ to an algebraic space. Then there exists a good quotient $\pi : X \to Y$ with Y an algebraic space.*

Proof. If $Y = \mathcal{S}pec_Z f_*(\mathcal{O}_X)^G$, then the induced morphism $\pi : X \to Y$ is a good quotient by Proposition 4.16, Proposition 4.13 and Corollary 4.14. □

5. Affine cosets

We are now in the position to recover Matsushima's theorem:

Theorem 1.2. *Suppose $G \to S$ is an affine, linearly reductive group scheme and $H \subseteq G$ is a flat, finitely presented, separated subgroup scheme. Then the following are equivalent:*

(i) *$H \to S$ is linearly reductive;*

(ii) *$G/H \to S$ is affine;*
(iii) *the functor $\mathcal{F} \mapsto \mathrm{Ind}_H^G \mathcal{F}$ from $\mathrm{QCoh}^H(S)$ to $\mathrm{QCoh}^G(S)$ is exact.*

Proof. First note that, since $G \to S$ is linearly reductive, it follows that $BG \to S$ is cohomologically affine. Suppose $H \to S$ is linearly reductive, and so $BH \to S$ is cohomologically affine. Since G is affine, $G/H \to BH$ is also affine, and hence the composition $G/H \to BH \to S$ is cohomologically affine. By Serre's criterion (Remark 4.7(i)), $G/H \to S$ is therefore affine. Conversely, suppose $G/H \to S$ is affine. Consider the 2-cartesian square

$$\begin{array}{ccc} G/H & \longrightarrow & S \\ \downarrow & & \downarrow \\ BH & \longrightarrow & BG. \end{array}$$

By descent, the morphism $BH \to BG$ is affine. Therefore the composition $BH \to BG \to S$ is cohomologically affine, and so $H \to S$ is linearly reductive. This proves the equivalence of (i) and (ii).

Condition (iii) is a direct translation of the condition of cohomological affineness for the morphism $BH \to BG$, and is thus equivalent by descent and Serre's criterion (Remark 4.7(i)) to (ii). □

The following result was used by Białynicki-Birula ([BB63]) and Richardson ([Ric77] to prove Matsushima's theorem. A proof also appeared in [Lun73, p. 85]. The language of stacks provides a quick proof relying essentially only on descent for affine morphism.

Proposition 5.1 (cf. [BB63, Lemma 1])**.** *Let $G \to S$ be a flat, finitely presented, separated group scheme, and $H_2 \subseteq H_1$ be an inclusion of flat, finitely presented, separated subgroup schemes of G, with H_1 quasi-affine over S. Then $H_1/H_2 \to S$ is affine if and only if $G/H_2 \to G/H_1$ is affine.*

Proof. The 2-cartesian diagrams

$$\begin{array}{ccc} H_1/H_2 & \longrightarrow & S \\ \downarrow & & \downarrow \\ BH_2 & \longrightarrow & BH_1 \end{array} \qquad \begin{array}{ccccc} G/H_2 & \longrightarrow & G/H_1 & \longrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ BH_2 & \longrightarrow & BH_1 & \longrightarrow & BG \end{array}$$

and descent theory immediately imply the result. □

Corollary 5.2 (cf. [BB63, Corollary 1])**.** *Let $H_2 \subseteq H_1 \subseteq G$ be inclusions of group schemes as in Proposition 5.1. If H_1/H_2 and G/H_1 are affine over S, then so is G/H_2.*

6. Observable subgroups

For a subgroup $H \subseteq G$, Matsushima's theorem provides a relationship between the reductiveness of H and the affineness of G/H (see Theorem 1.2). In this section, we analyze the relationship between the observability of H (i.e., whether every H-representation is a subrepresentation of a G-representation) and the quasi-affineness of G/H. Our approach is in the same spirit as the proof of Theorem 1.2: we interpret the quasi-affineness of G/H in terms of functorial properties of $BH \to BG$. We first prove a characterization of quasi-affine morphisms generalizing [Gro67, II.5.1.2 and IV.5.1.2].

Consider the following property for a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks:

($\star$) *For any quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, the adjunction morphism $f^*f_*\mathcal{F} \to \mathcal{F}$ is surjective.*

Lemma 6.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact (and quasi-separated) representable morphism of algebraic stacks and $g : \mathcal{Y}' \to \mathcal{Y}$ a morphism of algebraic stacks. Consider the 2-cartesian product*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

Then the following hold:

(i) **(Descent)** *if f' satisfies ($\star$) and g is faithfully flat, then f satisfies ($\star$); and*
(ii) **(Base change)** *if f satisfies ($\star$) and $\mathcal{Y}$ has quasi-affine diagonal, then f' satisfies ($\star$).*

Proof. For (i), let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module and $\alpha : f^*f_*\mathcal{F} \to \mathcal{F}$ be the adjunction morphism. Consider the commutative diagram

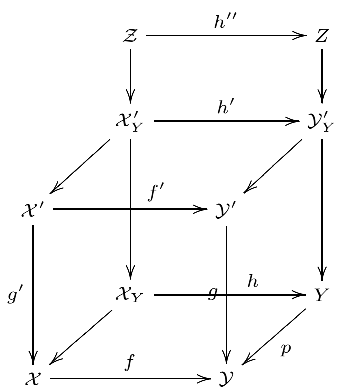
$$\begin{array}{ccc} g'^*f^*f_*\mathcal{F} & \xrightarrow{g'^*\alpha} & g'^*\mathcal{F} \\ \| & & \uparrow{\scriptstyle \alpha'} \\ f'^*g^*f_*\mathcal{F} & \xrightarrow{f'^*\psi} & f'^*f'_*g'^*\mathcal{F} \end{array}$$

where $\psi : g^*f_*\mathcal{F} \to f'_*g'^*\mathcal{F}$ and $\alpha' : f'^*f'_*(g'^*\mathcal{F}) \to g'^*\mathcal{F}$ are the natural adjunction maps. By flat base change, ψ is an isomorphism and hence $f'^*\psi$ is surjective. Since α' is also surjective, it follows that the composition $\alpha' \circ f'^*\psi = g'^*\alpha$ is surjective. Since g' is faithfully flat, we thus conclude α is surjective.

For (ii), we first show that the statement is true if either (a) g is a quasi-compact open immersion, or (b) g is an affine morphism. In case (a), we first note that since g and g' are open immersions, the adjunction morphisms of functors $g^*g_* \to \mathrm{id}$ and $g'^*g'_* \to \mathrm{id}$ are natural isomorphisms. Let $\mathcal{F}'$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}'}$-module. Then since f satisfies property ($\star$), $f^*f_*g'_*\mathcal{F}' \to g'_*\mathcal{F}'$ is surjective and therefore so is $f'^*g^*g_*f'_*\mathcal{F}' \cong g'^*f^*f_*g'_*\mathcal{F}' \to g'^*g'_*\mathcal{F}$. Since the adjunction morphisms are natural isomorphisms, this last morphism is canonically identified with the adjunction map $f'^*f'_*\mathcal{F}' \to \mathcal{F}'$. In case (b), since g' is affine, $f^*f_*\mathcal{F} \to \mathcal{F}$ is surjective if and only $g'_*f^*f_*\mathcal{F} \to g'_*\mathcal{F}$ is surjective. Since f satisfies property

$(\star)$, $f^* f_* g'_* \mathcal{F}' \to g'_* \mathcal{F}'$ is surjective, but this factors as $f^* f_* g'_* \mathcal{F}' \cong f^* g_* f'_* \mathcal{F}' \to g'_* f'^* f'_* \mathcal{F}' \to g'_* \mathcal{F}'$. We thus conclude $g'_* f^* f_* \mathcal{F} \to g'_* \mathcal{F}$ is surjective.

Therefore, property $(\star)$ is always stable under quasi-affine base change. We now prove the statement for arbitrary g (when $\delta_{\mathcal{Y}/S} : \mathcal{Y} \to \mathcal{Y} \times_S \mathcal{Y}$ is quasi-affine). Since the question is Zariski-local, we may assume the base scheme S is affine and that $\mathcal{Y}$ and $\mathcal{Y}'$ are quasi-compact. Let $p : Y \to \mathcal{Y}$ be a smooth presentation with Y affine. Since $\Delta_{\mathcal{Y}/S}$ is quasi-affine, $Y \times_{\mathcal{Y}} Y \cong \mathcal{Y} \times_{\mathcal{Y} \times_S \mathcal{Y}} (Y \times_S Y)$ is quasi-affine and $p : Y \to \mathcal{Y}$ is a quasi-affine morphism. After base changing by $p : Y \to \mathcal{Y}$ and choosing a smooth presentation $Z \to \mathcal{Y}'_Y := \mathcal{Y}' \times_{\mathcal{Y}} Y$ with Z an affine scheme, we have the 2-cartesian diagram:



Since f satisfies property $(\star)$ and p is a quasi-affine morphism, by our above argument h satisfies property $(\star)$. The morphism $Z \to Y$ is affine which implies that h'' satisfies property $(\star)$. Since the composition $Z \to \mathcal{Y}'_Y \to \mathcal{Y}'$ is smooth and surjective, it follows by descent that f' satisfies property $(\star)$. $\square$

Proposition 6.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact (and quasi-separated) representable morphism of algebraic stacks, with $\mathcal{Y}$ having quasi-affine diagonal. Then the following are equivalent:*

(i) *f is quasi-affine;*

(ii) *for any quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, the adjunction morphism $f^* f_* \mathcal{F} \to \mathcal{F}$ is surjective; and*

(iii) *for any quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, there exists a quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{G}$ and a surjection $f^* \mathcal{G} \to \mathcal{F}$;*

If, in addition, $\mathcal{X}$ and $\mathcal{Y}$ are locally noetherian, then the above conditions are also equivalent to:

(iv) *for any coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, the adjunction morphism $f^* f_* \mathcal{F} \to \mathcal{F}$ is surjective; and*

(v) *for any coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, there exists a coherent $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{G}$ and a surjection $f^* \mathcal{G} \to \mathcal{F}$.*

Proof. First, it is clear (ii) implies (iii). To see (iii) implies (ii), suppose $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-module and $f^* \mathcal{G} \to \mathcal{F}$ is a surjection. The counit of the adjunction then gives a factorization $f^*\mathcal{G} \to f^*f_*\mathcal{F} \to \mathcal{F}$, and hence $f^*f_*\mathcal{F} \to \mathcal{F}$ is also surjective.

We next show (i) is equivalent to (ii). Note that the property of being quasi-affine is stable under composition and base change and descends under faithfully flat morphisms on the target. By Lemma 6.1, for a quasi-compact (and quasi-separated) representable morphism of algebraic stacks, property (ii) is also stable under composition and descends under faithfully flat morphisms on the target. Therefore, in showing the equivalence between (i) and (ii), we can reduce to the case where $f : X \to Y$ is a quasi-compact (and quasi-separated) morphism of algebraic spaces with Y an affine scheme. As open immersions and affine morphisms satisfy property (ii), quasi-affine morphisms also do. Thus, (i) implies (ii).

To see that (ii) implies (i) we may suppose that $f : X \to Y = \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$. If $\mathcal{I} \subseteq \mathcal{O}_X$ is a sheaf of ideals, then $f_*\mathcal{I} \subseteq f_*\mathcal{O}_X = \mathcal{O}_Y$ is a sheaf of ideals. It follows that for a closed subset $Z \subseteq X$, $f^{-1}(\overline{f(Z)}) = Z$. In particular, f is injective.

We first claim that, if Y is a local scheme (i.e., $\Gamma(X, \mathcal{O}_X))$ is a local ring) in which the unique closed point $y \in Y$ is the image of a closed point $x \in X$, then $X = Y$. First note that by the previous paragraph, we know that $f^{-1}(y) = \{x\}$, and so x is the unique closed point of X. Choose a finite surjective morphism $p : U \to X$ from a scheme U (which exists by [LMB00, Theorem 16.6], [Ryd10b, Theorem B]). Then U is quasi-affine (by the result for schemes, [Gro67, II.5.1.2, IV.1.7.16]) and U only has finitely many closed points. We now claim that it follows that U is affine. Let $u_1, \ldots, u_n$ be the closed points of U and $i : U \hookrightarrow Z = \operatorname{Spec} \Gamma(U, \mathcal{O}_U)$ be the open immersion. If $U \neq Z$, there exists a closed point $z \in Z \setminus U$. Then there is a function $f \in \Gamma(U, \mathcal{O}_U)$ such that $f(z) = 0$ but $f(u_i) \neq 0$: otherwise $\mathfrak{p}_z \subseteq \bigcup_i \mathfrak{p}_{u_i}$ where $\mathfrak{p}_z, \mathfrak{p}_{u_1}, \ldots, \mathfrak{p}_{u_n}$ denote the prime ideals of $\Gamma(X, \mathcal{O}_X)$ defined by $z, u_1, \ldots, u_n$ which implies that $\mathfrak{p}_z \subseteq \mathfrak{p}_{u_i}$ for some i, a contradiction. But then f is an invertible function on U and therefore on Z, contradicting $f(z) = 0$. Therefore U is affine and by Chevalley's Theorem ([Knu71, Theorem III.4.1], [Ryd10b, Theorem 8.1]) X is also affine and $X = Y$.

For general affine Y, let $x \in X$ be a point and $y = f(x)$. Consider the base change by $\operatorname{Spec} \mathcal{O}_{Y,y} \to Y$. We can write $\operatorname{Spec} \mathcal{O}_{Y,y} = \varprojlim Y_i$ with $Y_i \subseteq Y$ affine open subschemes. We have a diagram

$$\begin{array}{ccccc}
X & \longleftarrow & X \times_Y Y_i & \longleftarrow & X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y} \\
\downarrow{\scriptstyle f} & & \downarrow & & \downarrow \\
Y & \hookleftarrow & Y_i & \longleftarrow & \operatorname{Spec} \mathcal{O}_{Y,y}
\end{array}$$

By the above case for a local scheme, we know that

$$X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y} = \varprojlim \, (X \times_Y Y_i)$$

is affine. By [Ryd10b, Theorem C], it follows that for $i >> 0$, $X \times_Y Y_i$ is affine. In particular, $x \in X$ has an open neighborhood which is a scheme. Therefore, X is a scheme and the result follows from the schematic version of the statement.

The fact that (ii) implies (iv) is obvious. For the converse, let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module. Using [LMB00, Theorem 15.4], we can write $\mathcal{F} = \varinjlim \mathcal{F}_i$ as a filtered direct limit of coherent subsheaves $\mathcal{F}_i \subseteq \mathcal{F}$. For each i, there is a commutative diagram

$$\begin{array}{ccc} f^* f_* \mathcal{F}_i & \longrightarrow & \mathcal{F}_i \\ \downarrow & & \downarrow \\ f^* f_* \mathcal{F} & \longrightarrow & \mathcal{F}. \end{array}$$

If $s \in \Gamma(\operatorname{Spec} A \to \mathcal{X}, \mathcal{F})$ is a section of $\mathcal{F}$ over a smooth morphism $\operatorname{Spec} A \to \mathcal{X}$, then there exists j such that $s \in \Gamma(\operatorname{Spec} A \to \mathcal{X}, \mathcal{F}_j)$. Since $f^* f_* \mathcal{F}_j \to \mathcal{F}_j$ is surjective, s is in the image of $\Gamma(\operatorname{Spec} A \to \mathcal{X}, f^* f_* \mathcal{F}_j)$ and therefore in the image of $\Gamma(\operatorname{Spec} A \to \mathcal{X}, f^* f_* \mathcal{F})$.

The proof of the equivalence of (iv) and (v) is analogous to that of (ii) and (iii). □

Remark 6.3.

(i) By a different method, Philipp Gross has recently shown the same result. In fact, his result is more general in that he considers non-representable morphisms $f : \mathcal{X} \to \mathcal{Y}$ where the relative inertia $I_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ has affine fibers. Moreover, when $\mathcal{Y}$ does not have a quasi-affine diagonal, he considers the more natural property that $(\star)$ holds for all base changes. See [Gro10].

(ii) The assumption that $\mathcal{Y}$ have quasi-affine diagonal is necessary for (ii) to imply (i). For example, if $G \to \operatorname{Spec} k$ is any finite type group scheme, then $\operatorname{Spec} k \to BG$ satisfies (ii). Indeed, for any k-vector space V, the adjunction map corresponds to the map $V \otimes_k \Gamma(G, \mathcal{O}_G) \to V$, which is surjective. However, if $G \to \operatorname{Spec} k$ is an abelian variety, then $\operatorname{Spec} k \to G$ is not quasi-affine and hence does not satisfy (i).

Definition 6.4. Let $G \to S$ be a flat, finitely presented, quasi-affine group scheme. A flat, finitely presented, quasi-affine subgroup scheme $H \subseteq G$ is *observable* if every quasi-coherent $\mathcal{O}_S[H]$-module is a quotient of a quasi-coherent $\mathcal{O}_S[G]$-module.

Remark 6.5. If $S = \operatorname{Spec} k$, this is easily seen to be equivalent to the definition in [BBHM63]: a subgroup scheme $H \subset G$ is observable if every finite dimensional H-representation is a sub-H-representation of a finite dimensional G-representation.

We can now prove:

Theorem 1.3. *Let $G \to S$ be a flat, finitely presented, quasi-affine group scheme and $H \subseteq G$ a flat, finitely presented, quasi-affine subgroup scheme. The following are equivalent:*

(i) *H is observable;*
(ii) *for every quasi-coherent $\mathcal{O}_S[H]$-module $\mathcal{F}$, the counit morphism of the adjunction, $\operatorname{Ind}_H^G \mathcal{F} \to \mathcal{F}$, is a surjection of $\mathcal{O}_S[H]$-modules;*
(iii) *$BH \to BG$ is quasi-affine; and*
(iv) *$G/H \to S$ is quasi-affine.*

If S is noetherian, then the above are also equivalent to:

(v) *every coherent $\mathcal{O}_S[H]$-module is a quotient of a coherent $\mathcal{O}_S[G]$-module; and*

(vi) *for every coherent $\mathcal{O}_S[H]$-module $\mathcal{F}$, the counit morphism of the adjunction, $\operatorname{Ind}_H^G \mathcal{F} \to \mathcal{F}$, is a surjection of $\mathcal{O}_S[H]$-modules.*

Proof. The equivalences follow from the definitions and Proposition 6.2. We only add that since $S \to BG$ is faithfully flat and finitely presented and $G/H \cong BH \times_{BG} S$, descent implies $BH \to BG$ is quasi-affine if and only if $G/H \to S$ is quasi-affine. □

7. Existence of good moduli spaces

Given an algebraic stack $\mathcal{X}$, we would like local conditions that guarantee the existence of a good moduli space. Recall first the following definition:

Definition 7.1 ([Alp08, Def. 6.1])**.** If $\pi : \mathcal{X} \to Y$ is a good moduli space, an open substack $\mathcal{U} \subseteq \mathcal{X}$ is *saturated for* π if there is a set-theoretic equality $\pi^{-1}(\pi(\mathcal{U})) = \mathcal{U}$.

Saturated substacks have the following nice property (see [Alp08, Remark 6.2]): if an open substack $\mathcal{U} \subseteq \mathcal{X}$ is saturated for π, then $\pi(\mathcal{U})$ is open and $\pi|_{\mathcal{U}} : \mathcal{U} \to \pi(\mathcal{U})$ is a good moduli space. Moreover, we have the following proposition:

Lemma 7.2 ([Alp08, Prop. 7.9])**.** *Suppose $\mathcal{X}$ is a noetherian algebraic stack containing open substacks $\{\mathcal{U}_i\}_{i \in I}$ such that for each i there exists a good moduli space $\pi_i : \mathcal{U}_i \to Y_i$, with Y_i a scheme (resp., an algebraic space). Let $\mathcal{U} = \bigcup \mathcal{U}_i$. Then the following are equivalent:*

(i) *$\mathcal{U}_i \cap \mathcal{U}_j$ is saturated for π_i for every i, j; and*

(ii) *there exists a good moduli space $\pi : \mathcal{U} \to Y$ with Y a scheme (resp., an algebraic space), and algebraic subspaces $\widetilde{Y}_i \subseteq Y$ with $\widetilde{Y}_i \cong Y_i$ and $\pi^{-1}(\widetilde{Y}_i) = \mathcal{U}_i$.*

It would be useful to have an intrinsic definition of saturated that does not refer to a good moduli space, and to use this definition to find conditions that guarantee the existence of open covers by saturated substacks. Combined with the previous proposition, this would enable us to give local conditions guaranteeing the ability to glue local good moduli spaces. We first make the following definition:

Definition 7.3. Suppose $\mathcal{X}$ is an algebraic stack over a scheme S and $\mathcal{Z} \subseteq \mathcal{X}$ is a closed substack. Define $F_{\mathcal{X}}(\mathcal{Z}) \subseteq |\mathcal{X}|$ to be the set of those points $x \in |\mathcal{X}|$ for which there is a representative (equivalently, for all representatives; see Remark 7.4) $\bar{x} : \operatorname{Spec} k \to \mathcal{X}$ with $\overline{\{x\}} \cap (\mathcal{Z} \times_S k) \neq \varnothing$ in $|\mathcal{X} \times_S k|$. (Here we have abused notation by considering $x \in |\mathcal{X} \times_S k|$ as the image of $\operatorname{Spec} k \to \mathcal{X} \times_S k$.)

Remark 7.4.

(i) To verify the stated equivalence, let $\bar{x}_1 : \operatorname{Spec} k_1 \to \mathcal{X}$ and $\bar{x}_2 : \operatorname{Spec} k_2 \to \mathcal{X}$ be equivalent representatives of $x \in |\mathcal{X}|$, given by a field inclusion $k_1 \hookrightarrow k_2$. Then $\bar{x}_2 : \operatorname{Spec} k_2 \to \mathcal{X}$ is isomorphic in $\mathcal{X}(k_2)$ to the composition $\operatorname{Spec} k_2 \to \operatorname{Spec} k_1 \xrightarrow{\bar{x}_1} \mathcal{X}$. If $\overline{\{x\}}_i$ denotes the closure of the image of $\operatorname{Spec} k_i \to \mathcal{X} \times_S k_i$, then by flat base change $\overline{\{x\}}_2 = \overline{\{x\}}_1 \times_{k_1} k_2$. Therefore, $\overline{\{x\}}_1 \cap (\mathcal{Z} \times_S k_1) \neq \varnothing$ if and only if $\overline{\{x\}}_2 \cap (\mathcal{Z} \times_S k_2) \neq \varnothing$.

(ii) Note that if $F_{\mathcal{X}}(\mathcal{Z}) \subseteq |\mathcal{X}|$ is closed, then $F_{\mathcal{X}}(F_{\mathcal{X}}(\mathcal{Z})) = F_{\mathcal{X}}(\mathcal{Z})$.

Lemma 7.5. *Suppose $\pi : \mathcal{X} \to Y$ is a good moduli space. Then $|\pi^{-1}(\pi(\mathcal{Z}))| = F_{\mathcal{X}}(\mathcal{Z})$ for every closed substack $\mathcal{Z} \subseteq \mathcal{X}$. In particular, $F_{\mathcal{X}}(\mathcal{Z})$ is closed.*

Proof. Suppose $x \in \pi^{-1}(\pi(\mathcal{Z}))$, with representative $\operatorname{Spec} k \to \mathcal{X}$. If $s : \operatorname{Spec} k \to \mathcal{X} \to S$ denotes the composition, then the base change $\pi_s : \mathcal{X} \times_S k \to Y \times_S k$ is a good moduli space. It follows that $\pi_s(\overline{\{x\}}) \cap \pi_s(\mathcal{Z} \times_S k) \neq \varnothing$ in $Y \times_S k$. By [Alp08, Theorem 4.15(iii)], this implies $\overline{\{x\}} \cap (\mathcal{Z} \times_S k) \neq \varnothing$ in $|\mathcal{X} \times_S k|$, and hence $x \in F_{\mathcal{X}}(\mathcal{Z})$. Conversely, suppose $x \in F_{\mathcal{X}}(\mathcal{Z})$, with representative $\operatorname{Spec} k \to \mathcal{X}$, and choose any point $z \in \overline{\{x\}} \cap (\mathcal{Z} \times_S k) \subseteq |\mathcal{X} \times_S k|$. Then $\overline{\{x\}} \cap \overline{\{z\}} \neq \varnothing$ in $|\mathcal{X} \times_S k|$, and so by [Alp08, Theorem 4.15(iv)] we have $\pi_s(x) = \pi_s(z)$. It follows that $x \in \pi^{-1}(|\pi(\mathcal{Z})|)$. $\square$

Lemma 7.6. *Suppose $\pi : \mathcal{X} \to Y$ is a good moduli space. Let $\mathcal{U} \subseteq \mathcal{X}$ be an open substack with reduced complement $\mathcal{Z}$. Then the following are equivalent:*

(i) *$\mathcal{U}$ is saturated for π;*
(ii) *$F_{\mathcal{X}}(\mathcal{Z}) = \mathcal{Z}$;*
(iii) *for every point $u \in |\mathcal{U}|$ and representative $\bar{u} : \operatorname{Spec} k \to \mathcal{U}$ of u, the closure $\overline{\{u\}} \subseteq |\mathcal{X} \times_S k|$ is contained in $|\mathcal{U} \times_S k|$; and*
(iv) *for every point $u \in |\mathcal{U}|$ and representative $\bar{u} : \operatorname{Spec} k \to \mathcal{U}$ of u for which $u \in |\mathcal{U} \times_S k|$ is closed, $u \in |\mathcal{X} \times_S k|$ is also closed.*

Proof. These properties are all local on Y so we may assume that Y is quasi-compact; it then follows from the definition of a good moduli space (Definition 4.9) that $\mathcal{X}$ is also quasi-compact. First note that $\pi^{-1}(\pi(\mathcal{U})) = \mathcal{U}$ if and only if $\pi^{-1}(\pi(\mathcal{Z})) = \mathcal{Z}$. By Lemma 7.5, statements (i) through (iii) are equivalent. It is clear (iii) implies (iv). To prove the converse, suppose (by way of contradiction) there exists a closed point $x_0 \in \overline{\{u\}} \cap (\mathcal{Z} \times_S k) \subseteq |\mathcal{X} \times_S k|$. Let $s : \operatorname{Spec} k \to \mathcal{U} \to S$ be the composition and $\pi_s : \mathcal{X} \times_S k \to Y \times_S k$ be the base change. Then for any closed point $u_0 \in \overline{\{u\}} \subseteq |\mathcal{U} \times_S k|$, by [Alp08, Theorem 4.15] we have $\pi_s(u) = \pi_s(x_0) = \pi_s(u_0)$ and $x_0 \in \overline{\{u_0\}}$. In particular, $u_0 \in |\mathcal{U} \times_S k|$ is closed but $u_0 \in |\mathcal{X} \times_S k|$ is not closed, violating (iv). $\square$

Remark 7.7. If $\mathcal{X}$ is finite type over $S = \operatorname{Spec} k$ with k algebraically closed, then the equivalence of (i) and (iv) simply states that an open substack $\mathcal{U} \subseteq \mathcal{X}$ is saturated for π if and only if every closed point $u \in \mathcal{U}(k)$ is also closed in $\mathcal{X}$, i.e., the open immersion $\mathcal{U} \hookrightarrow \mathcal{X}$ maps closed points to closed points.

By Lemma 7.6, it is reasonable to make the following definition:

Definition 7.8. An open substack $\mathcal{U} \subseteq \mathcal{X}$ is *saturated* if for every point $u \in |\mathcal{U}|$ and representative $\bar{u} : \operatorname{Spec} k \to \mathcal{U}$ of u, the closure $\overline{\{u\}} \subseteq |\mathcal{X} \times_S k|$ is contained in $|\mathcal{U} \times_S k|$.

Remark 7.9. Note that $\mathcal{U} \subseteq \mathcal{X}$ is saturated if and only if $F_{\mathcal{X}}(\mathcal{Z}) = |\mathcal{Z}|$, where $\mathcal{Z} = \mathcal{X} \setminus \mathcal{U}$ is the reduced complement. See [Alp10b, Section 2] for more general notions of saturated and weakly saturated morphisms, as well as their properties.

As a converse to Lemma 7.5, we have the following lemma:

Lemma 7.10 (cf. [BBŚ, Lemma 1])**.** *Suppose the set $F_{\mathcal{X}}(\mathcal{Z}) \subseteq |\mathcal{X}|$ is closed for every closed substack $\mathcal{Z} \subseteq \mathcal{X}$. Then for every open substack $\mathcal{U} \subseteq \mathcal{X}$, there exists an open substack $\mathcal{V} \subseteq \mathcal{U}$ such that:*

(i) $\mathcal{V} \subseteq \mathcal{X}$ *is saturated; and*
(ii) *for every* $u \in |\mathcal{U}|$ *and representative* $\bar{u} : \operatorname{Spec} k \to \mathcal{U}$ *of* u *with* $\overline{\{u\}} \subseteq |\mathcal{X} \times_S k|$ *contained in* $|\mathcal{U} \times_S k|$*, one has* $u \in |\mathcal{V}|$*. (In particular, any point* $u \in |\mathcal{U}|$ *with representative* $\operatorname{Spec} k \to \mathcal{U}$ *which is closed in* $|\mathcal{X} \times_S k|$ *is contained in* $|\mathcal{V} \times_S k|$*.)*

Proof. Let $\mathcal{Z}$ be the reduced closed complement $\mathcal{X} \setminus \mathcal{U}$. Then $\mathcal{V} = \mathcal{X} \setminus F_{\mathcal{X}}(\mathcal{Z}) \subseteq \mathcal{X}$ is a saturated open substack contained in $\mathcal{U}$ with the desired property. □

Since we are interested in determining conditions guaranteeing the existence of a good moduli space, Lemmas 7.2, 7.5 and 7.10 suggest we should establish when the sets $F_{\mathcal{X}}(\mathcal{Z})$ are closed for all closed substacks $\mathcal{Z} \subseteq \mathcal{X}$. We first give conditions guaranteeing the sets $F_{\mathcal{X}}(\mathcal{Z})$ are constructible.

Lemma 7.11. *Suppose* $\mathcal{X}$ *is a noetherian algebraic stack for which there exists a locally quasi-finite, universally submersive morphism* $f : \mathcal{W} \to \mathcal{X}$ *from an algebraic stack* $\mathcal{W}$ *admitting a good moduli space. Then for every closed substack* $\mathcal{Z} \subseteq \mathcal{X}$*, the set* $F_{\mathcal{X}}(\mathcal{Z}) \subseteq \mathcal{X}$ *is constructible.*

Remark 7.12. Étale morphisms and finite morphisms are locally quasi-finite and universally submersive.

Proof. By Lemma 7.5, $F_{\mathcal{W}}(f^{-1}(\mathcal{Z})) \subseteq |\mathcal{W}|$ is closed. We claim that $f(F_{\mathcal{W}}(f^{-1}(\mathcal{Z})))$ $= F_{\mathcal{X}}(\mathcal{Z})$. The containment $\subseteq$ is clear. Conversely, if $x \in F_{\mathcal{X}}(\mathcal{Z})$ with representative $\bar{x} : \operatorname{Spec} k \to \mathcal{X}$, then there exists a specialization $x \rightsquigarrow x_0$ with $x_0 \in |\mathcal{Z} \times_S k|$. Since $\mathcal{W} \times_S k \to \mathcal{Z} \times_S k$ is quasi-finite and submersive, there is a specialization $w \rightsquigarrow w_0$ in $|\mathcal{W} \times_S k|$ over $x \rightsquigarrow x_0$. Furthermore, the field extension $k(x) \to k(w)$ is finite, which implies $w \in F_{\mathcal{W}}(f^{-1}(\mathcal{Z}))$ exactly when $\overline{\{w\}} \cap f^{-1}(\mathcal{Z}) \neq \varnothing$. Therefore, $w \in F_{\mathcal{W}}(f^{-1}(\mathcal{Z}))$ and $x \in f(F_{\mathcal{W}}(f^{-1}(\mathcal{Z})))$. □

Lemma 7.13. *Let* $\mathcal{X}$ *be an algebraic stack of finite type over an algebraically closed field* k*. Suppose that:*

(i) $\mathcal{X} \cong [X/G]$*, where* X *is a scheme and* G *is a connected algebraic group; and*
(ii) *stabilizers of closed points in* $\mathcal{X}$ *are linearly reductive.*

Then for every closed substack $\mathcal{Z} \subseteq \mathcal{X}$*, the set* $F_{\mathcal{X}}(\mathcal{Z}) \subseteq \mathcal{X}$ *is constructible.*

Proof. If X is smooth, the statement follows from [Alp10b, Theorem 3]) and Lemma 7.11. We may reduce the normal case to the smooth case by applying Sumihiro's theorem ([Sum74]) which states that there exist Zariski-local G-equivariant embeddings of X into a projective scheme. The general case follows since normalization is finite and, in particular, quasi-finite and universally submersive. □

Remark 7.14. It appears that in the proof of [BBŚ, Lemma 2], the constructibility of $\{x \in X \mid Gx \cap Y \neq \varnothing\}$ is not verified for the action of the reductive group G on the algebraic variety X. It is checked that the set is closed under specialization, but one needs constructibility of the set to then conclude it is closed.

Lemma 7.15 (cf. [BBŚ, Lemma 2b])**.** *Suppose* $\mathcal{X}$ *satisfies the hypotheses of Lemma 7.13. Suppose further that either:*

(i) *for every smooth curve C over k and morphism $f : C \to \mathcal{X}$, the scheme-theoretic image of f admits a good moduli space; or*

(ii) *for every pair of points $x, y \in |\mathcal{X}|$, there exists an open substack $\mathcal{U}_{xy} \subseteq \mathcal{X}$ that contains x and y and admits a good moduli space.*

Then for every closed substack $\mathcal{Z} \subseteq \mathcal{X}$, the set $F_{\mathcal{X}}(\mathcal{Z}) \subseteq \mathcal{X}$ is closed.

Proof. By Lemma 7.11, the sets $F_{\mathcal{X}}(\mathcal{Z})$ are constructible, so it suffices to check that $F_{\mathcal{X}}(\mathcal{Z})$ is closed under specialization. First, suppose condition (i) holds. If $F_{\mathcal{X}}(\mathcal{Z})$ is not closed for some closed substack $\mathcal{Z} \subseteq \mathcal{X}$, then there exists a smooth pointed curve (C, p) and a morphism $f : C \to \mathcal{X}$ such that $f(C \setminus p) \subseteq F_{\mathcal{X}}(\mathcal{Z})$ but $f(p) \notin F_{\mathcal{X}}(\mathcal{Z})$. By assumption, the scheme-theoretic image $\mathcal{Y} \subseteq \mathcal{X}$ of f admits a good moduli space. But then Lemma 7.5 implies $F_{\mathcal{X}}(\mathcal{Z}) \cap \mathcal{Y} = F_{\mathcal{Y}}(\mathcal{Z} \cap \mathcal{Y})$ is closed, a contradiction.

Now instead suppose condition (ii) holds. If $F_{\mathcal{X}}(\mathcal{Z})$ is not closed for some closed substack $\mathcal{Z} \subseteq \mathcal{X}$, then there exists a closed point $x \in \overline{F_{\mathcal{X}}(\mathcal{Z})} \setminus F_{\mathcal{X}}(\mathcal{Z})$ that is a specialization of $x' \in F_{\mathcal{X}}(\mathcal{Z})$. By assumption, there are finitely many points $y_1, \dots, y_k \in |\mathcal{X}|$ and open substacks $\mathcal{U}_{xy_i}$ containing x and y_i, such that $\mathcal{U}_{xy_i}$ admits a good moduli space and $\bigcup_i \mathcal{U}_{xy_i} = \mathcal{X}$. Note that $F_{\mathcal{U}_{xy_i}}(\mathcal{X} \cap \mathcal{U}_{xy_i}) \subseteq |\mathcal{U}_{xy_i}|$ is closed and $F_{\mathcal{X}}(\mathcal{Z}) = \bigcup_i F_{\mathcal{U}_{xy_i}}(\mathcal{X} \cap \mathcal{U}_{xy_i})$. But $x' \in F_{\mathcal{U}_{xy_i}}(\mathcal{Z} \cap \mathcal{U}_{xy_i})$ for some i, which contradicts $x \notin F_{\mathcal{X}}(\mathcal{Z})$. □

Remark 7.16. In (ii) above, if were were instead to require the weaker property that every point have an open neighborhood admitting a good moduli space, then the conclusion would no longer hold. Consider, for example, the stack $\mathcal{X} = [\mathbf{P}^1/\mathbf{G}_m]$, where $\mathbf{G}_m$ acts by multiplication. In this case, $F_{\mathcal{X}}(\{0\}) = [(\mathbf{P}^1 \setminus \{\infty\})/\mathbf{G}_m]$ is not closed.

Proposition 7.17. *Suppose $\mathcal{X}$ is a noetherian algebraic stack such that:*

(i) *every point $x \in |\mathcal{X}|$ has an open neighborhood admitting a good moduli space; and*

(ii) *for every closed substack $\mathcal{Z} \subseteq \mathcal{X}$, $F_{\mathcal{X}}(\mathcal{Z}) \subseteq |\mathcal{X}|$ is closed.*

Then $\mathcal{X}$ admits a good moduli space.

Proof. For a closed point $x \in |\mathcal{X}|$, let $\mathcal{U}_x$ be an open neighborhood admitting a good moduli space. By Lemma 7.10, there exists an open neighborhood $\mathcal{V}_x \subseteq \mathcal{U}_x$ containing x such that $\mathcal{V}_x \subseteq \mathcal{X}$ is saturated. It follows also that $\mathcal{V}_x \subseteq \mathcal{U}_x$ is saturated, and so $\mathcal{V}_x$ also admits a good moduli space. For any pair of points $x, y \in |\mathcal{X}|$, $\mathcal{V}_x \cap \mathcal{V}_y$ is saturated in $\mathcal{V}_x$. Indeed, suppose $v \in |\mathcal{V}_x \cap \mathcal{V}_y|$ is closed but admits a specialization $v \rightsquigarrow v_0$ in $|\mathcal{V}_x|$. Since $\mathcal{V}_y \subseteq \mathcal{X}$ is saturated, we have $v_0 \in |\mathcal{V}_y|$, so that $v_0 \in |\mathcal{V}_x \cap \mathcal{V}_y|$ and $v = v_0$. By Lemma 7.2, the good moduli spaces of $\mathcal{V}_x$ can be glued to construct a good moduli space of $\mathcal{X}$. □

The proof of the following theorem now follows directly from Lemma 7.15 and Proposition 7.17.

Theorem 1.4. *Let G be a connected algebraic group acting on a scheme X of finite type over an algebraically closed field k, and suppose that for every pair of points $x, y \in X$, there exists a G-invariant open subscheme $U_{xy} \subseteq X$ that contains x and y and admits a good quotient. Then X admits a good quotient.*

References

[Alp08] J. Alper, *Good moduli spaces for Artin stacks*, `math.AG/0804.2242` (2008).

[Alp10a] J. Alper, *Adequate moduli spaces and geometrically reductive group schemes*, `math.AG/1005.2398` (2010).

[Alp10b] J. Alper, *On the local quotient structure of Artin stacks*, J. Pure Appl. Algebra **214** (2010), no. 9, 1576–1591.

[Art74] M. Artin, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165–189.

[BB63] A. Białynicki-Birula, *On homogeneous affine spaces of linear algebraic groups*, Amer. J. Math. **85** (1963), 577–582.

[BB02] A. Białynicki-Birula, *Quotients by actions of groups*, in: *Algebraic Quotients. Torus Actions and Cohomology. The Adjoint Representation and the Adjoint Action*, Encyclopaedia of Mathematical Sciences, Vol. 131, Springer, Berlin, 2002, pp. 1–82.

[BBHM63] A. Białynicki-Birula, G. Hochschild, G. D. Mostow, *Extensions of representations of algebraic linear groups*, Amer. J. Math. **85** (1963), 131–144.

[BBŚ] A. Białynicki-Birula, J. Święcicka, *Three theorems in existence of good quotients*, Math. Ann. **307** (1997), no. 1, 143–149.

[DM69] P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 75–109.

[Fan01] B. Fantechi, *Stacks for everybody*, European Congress of Mathematics, Vol. 1 (Barcelona, 2000), Progress in. Mathematics, Vol. 201, Birkhäuser, Basel, 2001, pp. 349–359.

[Gro67] A. Grothendieck, *Éléments de Géométrie Algébrique*, Inst. Hautes Études Sci. Publ. Math. (1961–1967), nos. 4, 8, 11, 17, 20, 24, 28, 32.

[Gro10] P. Gross, *Vector Bundles as Generators on Schemes and Stacks*, Ph.D. thesis, Heinrich-Heine-Universität Düsseldorf, 2010.

[Hab78] W. J. Haboush, *Homogeneous vector bundles and reductive subgroups of reductive algebraic groups*, Amer. J. Math. **100** (1978), no. 6, 1123–1137.

[Knu71] D. Knutson, *Algebraic Spaces*, Lectures Notes in Mathematics, Vol. 203, Springer-Verlag, Berlin, 1971.

[Kol97] J. Kollár, *Quotient spaces modulo algebraic groups*, Ann. of Math. (2) **145** (1997), no. 1, 33–79.

[LMB00] L. Gérard, L. Moret-Bailly, *Champs Algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics Vol. 39, Springer-Verlag, Berlin, 2000.

[Lun73] D. Luna, *Slices étales*, Bull. Soc. Math. France, Mémoire 33, 1973, 81–105.

[Mat60] Y. Matsushima, *Espaces homogènes de Stein des groupes de Lie complexes*, Nagoya Math. J. **16** (1960), 205–218.

[Mum65] D. Mumford, *Geometric Invariant Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34, Springer-Verlag, Berlin, 1965.

[Ric77] R. W. Richardson, *Affine coset spaces of reductive algebraic groups*, Bull. London Math. Soc. **9** (1977), no. 1, 38–41.

[Ryd10a] D. Rydh, *Étale devissage, descent and push-outs of stacks*, J. Algebra **311** (2011), no. 1, 194–223.

[Ryd10b] D. Rydh, *Noetherian approximation of algebraic spaces and stacks*, `math.AG/0904.0227v2` (2010).

[Ryd11] D. Rydh, *Existence and properties of geometric quotients*, `math.AG/0708.3333v1` (2011).

[Ses63] C. S. Seshadri, *Some results on the quotient space by an algebraic group of automorphisms*, Math. Ann. **149** (1962/1963), 286–301.

[spa] The stacks project authors, *Stacks project*, `http://math.columbia.edu/algebraic_geometry/stacks-git`.

[Sum74] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28.

[Vis89] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), 613–670.