Using Algebraic Geometry to Study the Motions of a Robotic Arm

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Abstract
In this study we summarize selected sections of David Cox, John Little, and Donal O’Shea’s *Ideals, Varieties, and Algorithms* and examine two major problems in robotics. Given the settings of the joints of a robot arm, what are the position and orientation of its hand? Is a given hand position and orientation possible, and, if so, what joint settings are needed? The first question is known as the *Forward Kinematic Problem*, and the second question is known as the *Inverse Kinematic Problem*. The first problem is more straightforward, though calculations quickly become lengthy as more joints of varying kind are added to a robot arm. The second problem is more abstract and there is room for more than one joint configuration for some given hand position and orientation.

1 Introduction

This project involved taking some concepts from algebraic geometry and applying them to solve problem in the field of robotics. In particular we looked at the forward kinematic problem and the inverse kinematic problem. The robots we considered in this study consisted of straight segments connected by joints. As a convention, when it comes to the ordering of segments and joints in the series of segments and joints that make up a robot arm, the fixed base segment is the first segment, and the hand is the last segment. The joints follow similarly.

1.1 Joint types

Following are descriptions with illustrations of the types of joints explored in this study.

1.1.1 Planar revolute joints

Planar revolute joints allow a planar change in angle between two joints. In other words, consider two segments connected by a planar revolute joint. Revolution about the axis joint always leaves the segments in the same plane no matter the angle of rotation. See the figure below.
1.1.2 Prismatic joints

Prismatic joints allow extending and shortening of the latter segment connected to the joint. In other words, if segment $i$ is closer to the base segment, and segment $i+1$ is connected to segment $i$ by a prismatic joint, then segment $i+1$ can change its length within a specified range. See the figure below.

Figure 2: An example prismatic joint joining segment $i$ and segment $i+1$; the length of segment $i+1$ is governed by how far the joint is extended
1.1.3 Ball joints

A ball joint allows motion in three dimensions. Fix the initial segment of a ball joint and consider the segment following. The segment following can rotate about the axis along the initial segment, and it can change its angle with the axis extending from the length of the initial segment. See the figure below.

Figure 3: An example ball joint joining segment $i$ and segment $i+1$ with angle $\theta_i$ off the $x_i$-axis and rotational angle $\rho_i$ around the $x_i$-axis.

1.1.4 Screw joints

A screw joint allows the following segment to change length by screwing or unscrewing along a thread on the initial segment. In this way, any subsequent segments will rotate about the axis down the length of the screw joint. Note that this rotation also results in a change in length of the second segment. See the figure below.

Figure 4: An example screw joint with angle $\theta_i$ connecting segment $i$ and segment $i+1$. 


1.1.5 Spin joints

The second segment in a spin joint is connected at a right angle to the axis down the length of the initial segment. The second segment then can rotate around this axis sweeping out a disk. See the figure below.

![Figure 5: An example spin joint with angle $\theta_i$ connecting segment $i$ and segment $i+1$](image)

Screw joints, ball joints, and spin joints all cause three-dimensional movement, while planar revolute and prismatic joints cause two-dimensional movement. In some of the examples below we consider robots with motion in three dimensions while in others we only consider planar robots whose motions are restricted to two dimensions. We do not consider physical attributes of the robot such as thickness of the segments nor the mass of segments. We allow all planar revolute joints to revolve fully around their axis of rotation.

1.1.6 Joint space and configuration space

We consider the set of possible configurations of joints. For example, in a planar revolute joint this is the set of all possible angles between the two segments it joins. Each robot we consider has a hand, which is the terminal segment of the robot which has a position and orientation. We may think of all the robots in these exercises as robot arms with the first segment fixed at a base, with one or more segments connected to it in series by the various joint types, terminated by the hand segment. We call the set of possible joint configurations the joint space.
and the set of possible hand positions and orientations with respect to the robot arm’s base the configuration space or operational space. We will see an example of finding the joint and configuration spaces of a robot arm in subsection 2 of section 2.

In these terms, the forward kinematic problem looks for a function \( f : J \rightarrow C \), where \( J \) is the joint space and \( C \) is the configuration space. The inverse kinematic problem looks for the possible joint settings \( x \) such that \( f(x) = c \) for some \( c \in C \). The configuration space \( C \) can be thought of as a Cartesian product between \( U \) and \( V \), where \( U \) is a subspace of the Euclidean space where the robot exists, and the elements of \( U \) correspond to the possible positions of the base of the hand segment of the robot; \( V \) is the possible orientations of the hand in space and is equal to \( S^1 \), the unit circle, or \( S^2 \), the unit sphere, depending on whether the robot under consideration is planar or has motion in three dimensions, respectively.

For this project we will look at how the different joints types affect the forward kinematic problem. For our investigation into the inverse kinematic problem, we will look exclusively at a planar robot arm with four segments connected by three planar revolute joints with the first segment being the base and the terminal segment being the hand.
2 The Forward Kinematic Problem

In this section we will explore the forward kinematic problem described previously. To reiterate, it is this: given the configuration of the joints of a robot, what is the position and orientation of its hand? We will first go over common labeling conventions so we may agree on a starting point for our computations. Next we will take a closer look at joint and configuration spaces and how they are defined. Finally we will compute $f$, the formula for the hand position and orientation given the joint configurations.

2.1 Labeling conventions

The following three figures illustrate the labeling convention used throughout this project. The first of the three shows how the base of a robot arm is labeled. Segment 1 of any robot arm is fixed and the $(x_1, y_1)$-coordinate system is based at the tip of segment 1. The x-axis is parallel to the floor, and the y-axis comes out along segment 1.

The point of the second diagram is to illustrate how subsequent segments are labeled. The $i^{th}$ segment is followed by the $i^{th}$ joint. The $i^{th}$ x-axis goes along the $i^{th}$ segment stemming at the $i-1^{th}$ joint. The $i^{th}$ y-axis is perpendicular to the $i^{th}$ x-axis and is 90 degrees counterclockwise from the $i^{th}$ x-axis. The $i^{th}$ angle is measured from the $x_i$-axis counterclockwise to the $x_{i+1}$-axis.
The last of these three diagrams shows an example of a fully labeled robot arm with three segments joined by two planar revolute joints. Note how the $x_1$-axis is parallel to the floor. This standard convention means that when this robot arm is in the *neutral* position, or when all the joints are configured to 0, the robot arm lies parallel to the floor.

Next we will take a more detailed look at how to specify joint and configuration spaces.
2.2 Joint spaces and configuration spaces

In this subsection, we will work through an exercise to see how to specify joint spaces and configuration spaces.

Exercise 1. Describe the joint space $J$ and the configuration space $C$ for the robot in Figure 9.

![Figure 6: The robot arm for Exercise 1](image)

Solution. The joint space $J$ is the set of all possible joint configurations, so for joints 1, 2, 3, and 5, the possible joint configurations range between 0 and $2\pi$ radians. The length of joint 4 ranges from 0 to $l_4$. Put this all together and we see that the joint space is $J = [0, 2\pi]^4 \times [0, l_4]$.

The configuration space is complicated, since we are not given the lengths of the arms. Let $a$ be the minimum distance between the hand and the origin (a depends on the lengths of the segments). Then let $U = \{(x, y) \in \mathbb{R}^2 : a \leq x^2 + y^2 \leq \sum_{i=2}^{5} l_i\}$, and let $V = [0, 2\pi]$ be the possible values that $\theta_5$ can take on. The configuration space is then $U \times V$.

2.3 Finding $f : J \rightarrow C$

We will now work through an exercise in finding a formula which takes in a joint configuration and returns a hand position and orientation.

Exercise 2. Consider a planar robot with a revolute joint 1, segment 2 of length $l_2$, a prismatic joint 2 with settings $l_3 \in [0, m_3]$, and a revolute joint 3, with segment 4 being the hand.

Construct an explicit formula for the mapping $f : J \rightarrow C$ in terms of trigonometric functions of the joint angles.
Solution. First let’s sketch a picture of the robot in question and label it as specified.

![Diagram of robot arm](image)

Figure 7: The robot arm for Exercise 2

Now we figure out the following:

- given a point \( P = (a_4, b_4) \) in the \((x_4, y_4)\)-coordinate system, what are its coordinates \((a_3, b_3)\) in the \((x_3, y_3)\)-coordinate system;

- given a point \((a_3, b_3)\) in the \((x_3, y_3)\)-coordinate system, what are its coordinates \((a_2, b_2)\) in the \((x_2, y_2)\)-coordinate system; and

- given a point \((a_2, b_2)\) in the \((x_2, y_2)\)-coordinate system, what are its coordinates \((a_1, b_1)\) in the \((x_1, y_1)\)-coordinate system.

To solve these problems we will use a rotation matrix. Notice that

\[
\begin{pmatrix}
  a_i \\
  b_i
\end{pmatrix} = \begin{pmatrix}
  \cos \theta_i & -\sin \theta_i \\
  \sin \theta_i & \cos \theta_i
\end{pmatrix} \cdot \begin{pmatrix}
  a_{i+1} \\
  b_{i+1}
\end{pmatrix} + \begin{pmatrix}
  l_i \\
  0
\end{pmatrix}
\]

This can be written more compactly in the following manner:

\[
\begin{pmatrix}
  a_i \\
  b_i \\
  1
\end{pmatrix} = \begin{pmatrix}
  \cos \theta_i & -\sin \theta_i & l_i \\
  \sin \theta_i & \cos \theta_i & 0 \\
  0 & 0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
  a_{i+1} \\
  b_{i+1} \\
  1
\end{pmatrix}.
\]

If we replace the matrix with \( A_i \), we have
\[
\begin{pmatrix}
  a_i \\
  b_i \\
  1
\end{pmatrix} = A_i \cdot \begin{pmatrix}
  a_{i+1} \\
  b_{i+1} \\
  1
\end{pmatrix}.
\]

We may iterate \(i\) from 4 back to 2 to get the following equation:

\[
\begin{pmatrix}
  a_1 \\
  b_1 \\
  1
\end{pmatrix} = A_1 \cdot A_2 \cdot A_3 \cdot \begin{pmatrix}
  a_4 \\
  b_4 \\
  1
\end{pmatrix}.
\]

If we do the matrix multiplication we have the following result:

\[
\begin{pmatrix}
  a_1 \\
  b_1 \\
  1
\end{pmatrix} = \begin{pmatrix}
  \cos (\theta_1 + \theta_2 + \theta_3) & -\sin (\theta_1 + \theta_2 + \theta_3) & l_3 \cos (\theta_1 + \theta_2) + l_2 \cos \theta_1 \\
  \sin (\theta_1 + \theta_2 + \theta_3) & \cos (\theta_1 + \theta_2 + \theta_3) & l_3 \sin (\theta_1 + \theta_2) + l_2 \sin \theta_1 \\
  0 & 0 & l_1
\end{pmatrix} \begin{pmatrix}
  a_4 \\
  b_4 \\
  1
\end{pmatrix}.
\]

Consider the fact that the hand lies at the origin of the \((x_4, y_4)\) system, so we can let \(a_4 = b_4 = 0\), note that \(l_3\) is variable, and realize that \(\theta_2 = 0\). Thus we have

\[
f(\theta_1, \theta_3, l_3) = \begin{pmatrix}
  (l_3 + l_2) \cos \theta_1 \\
  (l_3 + l_2) \sin \theta_1 \\
  \theta_1 + \theta_3
\end{pmatrix},
\]

where the first two components give us the \((x_1, y_1)\)-coordinates and we replace the last coordinate with the sum of the angles which determine the angle of the hand. Thus, we have \(f\).

\section{The Inverse Kinematic Problem}

In this section we will look at a new robot to investigate the inverse kinematic problem. It has four segments connected by three planar revolute joints. Segment two is 2 units long, and segment three is 1 unit long. For each \(y\) in the image of \(f\), we want to find the inverse image of \(y\), \(f^{-1}(y)\).
Here is $f$ for this robot arm:

$$f(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} \cos(\theta_1 + \theta_2) + 2 \cos \theta_1 \\ \sin(\theta_1 + \theta_2) + 2 \sin \theta_1 \\ \theta_1 + \theta_2 + \theta_3 \end{pmatrix}.$$  

Our goal here is to find the joint settings required to output a certain hand position and orientation. The orientation $\alpha$ of the hand is equal to $\theta_1 + \theta_2 + \theta_3$, so to achieve a certain $\alpha$, we look at the setting of $\theta_1$ and $\theta_2$ and adjust $\theta_3$ such that $\theta_3 = \alpha - (\theta_1 + \theta_2)$, hence it is easy to control the hand orientation.

The more difficult problem is to find the joint settings necessary for a certain hand position. Let $(a, b)$ be the desired hand position in the $(x_1, y_1)$ system. If we let $c_i = \cos \theta_i$ and $s_i = \sin \theta_i$, and use some trigonometric identities, we can write $a$ and $b$ in the following way:

$$a = c_1c_2 - s_1s_2 + 2c_1$$
$$b = c_1s_2 + c_2s_1 + 2s_1$$
$$0 = c_1^2 + s_1^2 - 1$$
$$0 = c_2^2 + s_2^2 - 1.$$  

We can compute a Groebner basis for the ideal generated by these four polynomials using Mathematica with the following command:

```math
polys = {c1*c2 - s1*s2 + 2 c1 - a, c1*s2 + c2*s1 + 2 s1 - b, c1^2 + s1^2 - 1, c2^2 + s2^2 - 1}

b = GroebnerBasis[polys, {c2, s2, c1, s1}, 
CoefficientDomain -> RationalFunctions]
```
If we do a little algebraic manipulation on the output, we find our basis to be

\[
s_1^2 - \frac{3b + a^2b + b^3}{2(a^2 + b^2)} s_1 + \frac{9 - 10a^2 + a^4 + 6b^2 + 2a^2b^2 + b^4}{16(a^2 + b^2)},
\]

\[
c_1 + \frac{b}{a} s_1 = \frac{3 + a^2 + b^2}{4a},
\]

\[
s_2 + \frac{a^2 + b^2}{a} - \frac{3b + a^2b + b^3}{4a},
\]

\[
c_2 + \frac{5 - (a^2 + b^2)}{4}.
\]

The variety of these polynomials gives the necessary joint configuration.

4 Conclusion

One aim of the problems studied here is to be able to give some guidance for the development of algorithms that can be used to plan the motion of a robot. Computer programs can be used to automatically generate a sequence of joint configurations that give the desired change from one point in the configuration space of a robot to another point. Since we have been dealing with idealized robots, we have not been concerning ourselves with physical constraints of a robot such as mass, joint strength, and the attributes of the motors which would actuate the movement of such a robot. In a real robot we would want to optimize the path of the robot hand and the necessary joint configurations to limit the joint movement speeds (which would affect mechanical wear and tear) and reduce the amount of total joint movement to achieve the desired path of the hand.

This study has given a glimpse into the problems motivated by robot movement. Further investigation would be necessary to develop a program for a particular robot. It would be interesting to come up with algorithms to find a path of joint configurations to achieve a change in hand configurations. The problems of robot kinematics will certainly play a role as robots are developed to help people in manufacturing, medicine, and other fields.