SENIOR PROJECT

TROPICAL CONICS AND THE SEARCH FOR A TROPICAL DUAL MAP

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Enumerative problems in algebraic geometry are those concerned with quantifying the number of geometric objects of a specified type possessing certain properties. For example, the following classical enumerative problem was posed by Jakob Steiner in 1848: "Given five conics in the plane, are there any conics tangent to all five? If so, how many are there?" [BKT08] provides a solution to Steiner's problem, as well as a solution to the following simpler problem: "Given p points and l lines in the plane, how many conics are there that contain the given points and are tangent to the given lines?"

In this paper we are not concerned with enumerative problems, but rather with a tool used to solve such problems. This tool, called "projective duality," was historically used to tackle so-called "excess intersections" that often arose in enumerative problems and that basic counting strategies could not handle. Duality in the complex projective plane $\mathbf{P}^2$ is a symmetry between points and lines, and has been known and exploited since the earliest days of algebraic geometry. As we will explain in Section 1, projective duality can be extended to a duality between projective plane conics. The goal of this project is to attempt to transport this classical projective duality to the newly developing field of tropical algebraic geometry.

Tropical algebraic geometry can roughly be thought of as the piecewise-linear geometry that results from applying a logarithmic map to a complex variety and then extracting the skeleton of the image. The linearity of the tropical objects is one of the great advantages of the theory, as it relates complicated algebro-geometric properties to problems in combinatorics and linear algebra which are often easier to study. This has led recently to an explosion of activity in the field, as algebraic geometers attempt to tropicalize all manner of classical constructions. As we will see, recreating classical structures in the tropical landscape is not always easy. The classic projective duality for plane conics can be interpreted in several different ways, and while we will attempt to tropicalize each, we will encounter serious difficulties in all of the constructions.

1 The Classical Dual Map

Before delving into projective duality, we first introduce the objects in which we are interested: conics in the projective plane. The equation for a general conic in $\mathbf{P}^2$ is given by

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0\,,$$

where $a, b, c, d, e, f \in \mathbf{C}$, not all zero. From this one equation we get three different kinds of conics: general (irreducible) conics, pairs of distinct lines, and double-line conics. **General conics** are those whose defining polynomial is irreducible, i.e., cannot be factored into the product of two linear polynomials. Such conics include circles, ellipses, hyperbolas, and parabolas. If the defining polynomial of a conic is reducible, the conic is said to be degenerate. A **pair of lines** is a degenerate conic whose defining polynomial factors as the product of two distinct linear polynomials. If the two linear polynomials happen to be the same, we call the pair of lines a **double line**.

In this section we demonstrate how to associate one conic to another conic (called its **dual**) and give several different interpretations of this dual map, with the hope that one of them will lead to a good tropical analogue.

1.1 The dual map for projective conics

One striking feature of the projective plane is a certain symmetry between points and lines, known as "projective duality." This duality has been studied and exploited to prove a great number of theorems over the years. It can also be extended to define a "duality" map on conics in the plane, as we explain below.

A general line l in $\mathbf{P}^2$ is given by an equation of the form $ax+by+cz=0$. Since multiplication by any nonzero scalar will not change the line, l can be represented by a point $[a:b:c]$ in another projective plane. We will call this point the **dual** to l, and denote it $\check{l}$. Observe that each line $l \subset \mathbf{P}^2$ corresponds to a unique point $[a:b:c]$, and conversely. We therefore have a bijection between lines and points in $\mathbf{P}^2$. Let us write $\check{\mathbf{P}}^2$ for the collection of lines in our original $\mathbf{P}^2$. We call $\check{\mathbf{P}}^2$ the **dual projective plane**, and identify it with a copy of $\mathbf{P}^2$ in the way described above.

Let $p=[x_0:y_0:z_0]$ be a point in $\mathbf{P}^2$. To each point in the plane we can consider the set of lines in the plane through that point. Applying the above construction to this set of lines, we obtain a set of collinear points in $\check{\mathbf{P}}^2$. That is, the dual to the point p is the line $\check{p}=\{[a:b:c]: x_0a+y_0b+z_0c=0\}$ in $\check{\mathbf{P}}^2$.

We can now describe a duality map on the conics in $\mathbf{P}^2$. Recall that a general conic in $\mathbf{P}^2$ is given by an equation of the form $ax^2+bxy+cy^2+dxz+eyz+fz^2=0$. To each point on the conic, there is a unique line in the plane passing through that point tangent to the conic. Mapping these lines in $\mathbf{P}^2$ to their corresponding points in $\check{\mathbf{P}}^2$, one can calculate using discriminants that the corresponding set of points in the dual projective plane are precisely the points $[x:y:z]$ that satisfy

$$(e^2-4cf)x^2+(4bf-2de)xy+(d^2-4af)y^2+(4cd-2be)xz+(4ae-2bd)yz+(b^2-4ac)z^2=0.$$

For a given general conic in $\mathbf{P}^2$ the above equation defines a conic in $\check{\mathbf{P}}^2$, which we call the **dual conic.** Note that while the above equation can be considered even for a non-general conic, it is possible for such degenerate conics that all of the coefficients in the above equation vanish. It can be shown algebraically that this happens exactly in the case of a double-line conic. One can also check that the dual to a pair of crossed lines $l_1 \cup l_2$ is the double-line conic through the two points $\check{l}_1$ and $\check{l}_2$.

1.2 Symmetric matrices for projective conics

The duality map for plane conics can be reinterpreted in terms of a duality map on symmetric 3×3 matrices, as follows. First recall that every homogeneous quadratic polynomial can be represented by a symmetric matrix. Suppose

$$F(x,y,z)=ax^2+bxy+cy^2+dxz+eyz+fz^2.$$

The symmetric matrix associated to F is

$$S=\begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}.$$

Computing the cofactor matrix of S, we obtain

$$\left[(-1)^{i+j}\det(S_{ij})\right]_{i,j}=\begin{bmatrix} cf-e^2/4 & -\left(fb/2-ed/4\right) & be/4-cd/2 \\ -\left(fb/2-de/4\right) & af-d^2/4 & -\left(ae/2-bd/4\right) \\ be/4-cd/2 & -\left(ae/2-bd/4\right) & ac-b^2/4 \end{bmatrix}.$$

The quadratic polynomial represented by this matrix is

$$\begin{aligned} G(x,y,z)=\left(cf-e^2/4\right)x^2+2\left(-\left(fb/2-ed/4\right)\right)xy+\left(af-d^2/4\right)y^2+2\left(be/4-cd/2\right)xz \\ +2\left(-\left(ae/2-bd/4\right)\right)yz+\left(ac-b^2/4\right)z^2. \end{aligned}$$

But look,

$$-4\cdot G(x,y,z)=(e^2-4cf)x^2+(4bf-2de)xy+(d^2-4af)y^2+(4cd-2be)xz+(4ae-2bd)yz+(b^2-4ac)z^2.$$

This is the same homogeneous quadratic we found in Section 1.1. That is, G defines the dual to the projective conic defined by F. Thus, using symmetric matrices, we have reconstructed our dual map for projective conics.

1.3 The blowup: extending the dual map to the double-line conics

Recall that a line $l \subset \mathbf{P}^2$ can be represented by the point in $\mathbf{P}^2$ corresponding to the coefficients of its defining equation. A conic $ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$ in $\mathbf{P}^2$ can be represented similarly by the point $[a : b : c : d : e : f] \in \mathbf{P}^5$. Using this, we can characterize the map from Section 1.1 in yet another way. Define a map $\delta : \mathbf{P}^5 \dashrightarrow \mathbf{P}^5$ by

$$[a : b : c : d : e : f] \mapsto [e^2 - 4cf : 4bf - 2de : d^2 - 4af : 4cd - 2be : 4ae - 2bd : b^2 - 4ac].$$

Note that δ is only a rational map, since it is not defined for points that correspond to double lines. These "bad" points, the set of all points in $\mathbf{P}^5$ corresponding to double lines, can be identified as the image of the so-called Veronese map. One way to correct this deficiency is to consider the closure of the graph of δ in $\mathbf{P}^5 \times \mathbf{P}^5$. This closure is known as the **blowup** of $\mathbf{P}^5$ along the Veronese surface V, and is denoted $Bl_V \mathbf{P}^5$. It is defined by the following system of twenty-three equations:

$$\begin{array}{ccc}
r(4bf - 2de) = s(e^2 - 4cf) & s(d^2 - 4af) = t(4bf - 2de) & t(4ae - 2bd) = v(d^2 - 4af) \\
r(d^2 - 4af) = t(e^2 - 4cf) & s(4cd - 2be) = u(4bf - 2de) & t(b^2 - 4ac) = w(d^2 - 4af) \\
r(4cd - 2be) = u(e^2 - 4cf) & s(4ae - 2bd) = v(4bf - 2de) & u(4ae - 2bd) = v(4cd - 2be) \\
r(4ae - 2bd) = v(e^2 - 4cf) & s(b^2 - 4ac) = w(4bf - 2de) & u(b^2 - 4ac) = w(4cd - 2be) \\
r(b^2 - 4ac) = w(e^2 - 4cf) & t(4cd - 2be) = u(d^2 - 4af) & v(b^2 - 4ac) = w(4ae - 2bd)
\end{array}$$

$$\begin{array}{cc}
bu + 2ew + 2cv = 0 & bv + 2dw + 2au = 0 \\
eu + 2br + 2cs = 0 & dv + 2bt + 2as = 0 \\
ds + 2et + 2fv = 0 & 4ar - 4ct + du - ev = 0 \\
es + 2dr + 2fu = 0 & 4ct - 4fw + bs - du = 0
\end{array}$$

The first fifteen equations above are precisely the equations one would expect the graph of δ to satisfy. The final eight equations, however, are so-called syzygies. Geometrically, they are the result of taking the closure of the graph. Algebraically, they were computed via computer software. Note that if $[a : b : c : d : e : f] \in \mathbf{P}^5$ is not on the Veronese surface, then the map is δ is well-defined. If $[a : b : c : d : e : f] \in \mathbf{P}^5$ *is* on the Veronese surface, however, then the first fifteen equations reduce to $0 = 0$, and the last eight equations define a $\mathbf{P}^2$ within $\mathrm{Bl}_V \mathbf{P}^5$. These are precisely the points picked up in closing the graph, and correspond to the extension of our duality map to the double-line conics. This exactly solves the problem of excess intersection in Steiner's problem.

2 Introduction to Tropical Algebraic Geometry

We now proceed with our main goal: to extend the duality map on projective plane conics to the tropical setting. We begin with a brief introduction to tropical numbers and the geometry in which we will be working.

2.1 Tropical algebra

The **tropical semiring** (sometimes called the **max-plus semiring**), is the set $\mathbf{T} = \mathbf{R} \cup \{-\infty\}$ endowed with the two operations

$$x \oplus y = \max(x, y), \qquad x \odot y = x + y.$$

We refer to these two operations as **tropical addition** and **tropical multiplication**, respectively. This arithmetic is roughly modeled on the arithmetic that results from applying a logarithmic map to a complex variety. To see this, recall the following two properties of logarithms:

$$\log(A + B) \approx \max(\log(A), \log(B)) \qquad \log(AB) = \log(A) + \log(B)$$

Note that $-\infty$ is the additive identity and zero is the multiplicative identity in the tropical semiring:

$$x \oplus -\infty = \max(x, -\infty) = x \qquad\qquad x \odot 0 = x + 0 = x$$

The tropical numbers have many properties in common with the usual real numbers. For example, tropical addition and tropical multiplication are both commutative and associative. However, observe that tropical addition is **idempotent**, i.e., $x \oplus x = x$ for every tropical number x. As a result, almost no tropical numbers have additive inverses. Indeed, for a tropical number x to have a tropical additive inverse, there would need to exist $y \in \mathbf{T}$ such that $x \oplus y = -\infty$. Clearly this is not possible unless $x = y = -\infty$. Thus tropical subtraction does not exist, which explains why $\mathbf{T}$ is only a semiring and not a ring.

2.2 Tropical polynomials and tropical curves

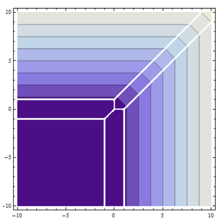
A **tropical polynomial** is an element of the semiring $\mathbf{T}[x_1, \ldots, x_n]$, i.e., an expression of the form

$$p(x_1, x_2, \ldots, x_n) = \bigoplus_i c_i \odot x_1^{\odot i_1} \odot x_2^{\odot i_2} \cdots \odot x_n^{\odot i_n} = \max_i(c_i + i_1 x_1 + i_2 x_2 + \ldots + i_n x_n).$$

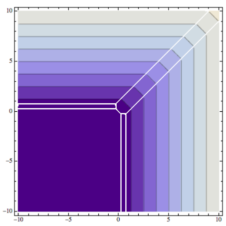
Every tropical polynomial also defines a piecewise-linear function $p : \mathbf{T}^n \to \mathbf{T}$, although it is important to note that different tropical polynomials can determine the same tropical function. A **tropical curve** in $\mathbf{T}^n$ is the corner locus of a tropical polynomial $p \in \mathbf{T}[x_1, \ldots, x_n]$. That is, a tropical curve is the locus of points in $\mathbf{T}^n$ for which the maximum of p is obtained at least twice. For this reason, corner loci are sometimes called **double-max loci**, or **bend loci**.

In this project we are particularly interested in tropical conics, which correspond to tropical quadrics in $\mathbf{TP}^2$. It turns out there are twenty types of tropical projective plane conics, and these types can be categorized based on their combinatorial properties; see [Z14]. Listed below are representative symmetric matrices for each of the twenty types, along with their corresponding double-max loci.

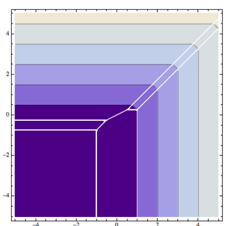
$$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



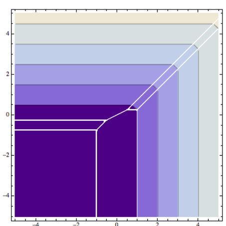
$$B_1 = \begin{bmatrix} 0 & 1 & {}^3\!/_4 \\ 1 & 0 & {}^3\!/_4 \\ {}^3\!/_4 & {}^3\!/_4 & 1 \end{bmatrix}$$



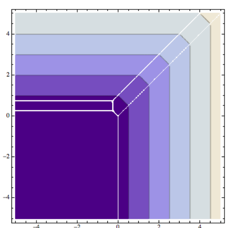
$$B_2 = \begin{bmatrix} 0 & {}^3\!/_4 & 1 \\ {}^3\!/_4 & 1 & {}^3\!/_4 \\ 1 & {}^3\!/_4 & 0 \end{bmatrix}$$



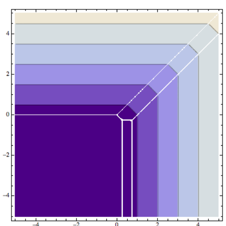
$$B_3 = \begin{bmatrix} 1 & {}^3\!/_4 & {}^3\!/_4 \\ {}^3\!/_4 & 0 & 1 \\ {}^3\!/_4 & 1 & 0 \end{bmatrix}$$



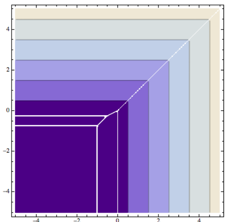
$$C_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & {}^3\!/_4 \\ 1 & {}^3\!/_4 & 1 \end{bmatrix}$$



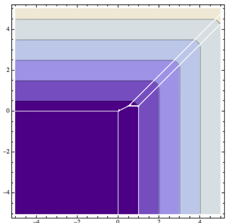
$$C_2 = \begin{bmatrix} 0 & 1 & {}^3\!/_4 \\ 1 & 1 & 1 \\ {}^3\!/_4 & 1 & 1 \end{bmatrix}$$



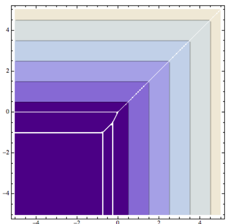
$$C_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & {}^{3}/_{4} \\ 1 & {}^{3}/_{4} & 0 \end{bmatrix}$$



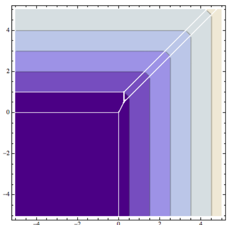
$$C_4 = \begin{bmatrix} 0 & {}^{3}/_{4} & 1 \\ {}^{3}/_{4} & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



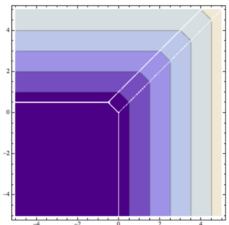
$$C_5 = \begin{bmatrix} 1 & 1 & {}^{3}/_{4} \\ 1 & 1 & 1 \\ {}^{3}/_{4} & 1 & 0 \end{bmatrix}$$



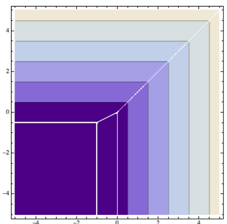
$$C_6 = \begin{bmatrix} 1 & {}^{3}/_{4} & 1 \\ {}^{3}/_{4} & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



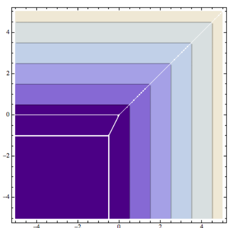
$$D_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & {}^{1}/_{2} \\ 1 & {}^{1}/_{2} & 1 \end{bmatrix}$$



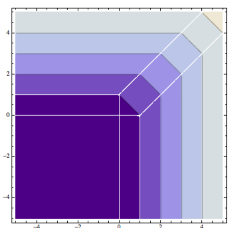
$$D_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & {}^{1}/_{2} \\ 1 & {}^{1}/_{2} & 0 \end{bmatrix}$$



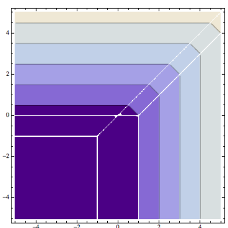
$$D_3 = \begin{bmatrix} 1 & 1 & {}^{1}/_{2} \\ 1 & 1 & 1 \\ {}^{1}/_{2} & 1 & 0 \end{bmatrix}$$



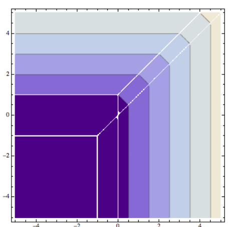
$$E_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



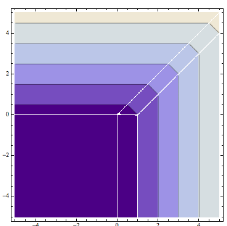
$$E_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



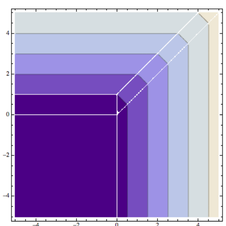
$$E_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



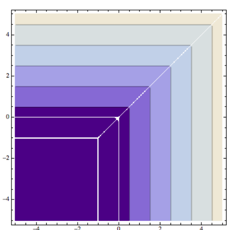
$$F_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



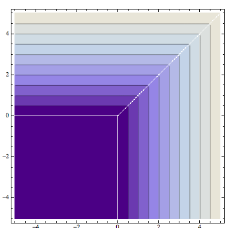
$$F_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



3 Finding a Tropical Dual Map for Tropical Conics

Now that we have established a foundation for tropical geometry, our goal is to extend the dual map on projective plane conics to a dual map on tropical conics. We attempt to tropicalize each of the classical constructions in turn.

3.1 Duality via tangent lines

Recall that in the classical case the original definition of the dual conic came from considering the set of lines in $\mathbf{P}^2$ tangent to the conic. Our first option for a tropical dual map analogous to the classical case is to consider mapping each conic to the locus of tropical lines tangent to the conic. This is difficult in the tropical setting, however, as there does not exist an established definition of tropical tangency. Indeed, it is not even obvious what "tangent" should mean in the tropical situation. There are many types of tropical conics with the property that every tropical line intersects the conic in exactly two points. Such conics would not have *any* tangent tropical lines. Because of this, we turn our attention away from tangent duality and towards symmetric matrices.

3.2 Duality via symmetric matrices

Similar to the matrices in Section 1.2, we can form the tropical symmetric matrix corresponding to a tropical conic's defining polynomial and propose a dual matrix. Given a tropical quadric

$$(a \odot x^{\odot 2}) \oplus (b \odot x \odot y) \oplus (c \odot y^{\odot 2}) \oplus (d \odot x \odot z) \oplus (e \odot y \odot z) \oplus (f \odot z^{\odot 2}),$$

the corresponding symmetric matrix is

$$S = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}.$$

Fortunately, we do not need to halve the coefficients in the tropical symmetric matrix like we did in the classical case since tropical addition is idempotent. While this algebra results in simpler arithmetic for finding the symmetric matrix corresponding to a general tropical conic, it also presents an obstacle to finding a dual matrix.

Because tropical subtraction does not exist, we cannot simply form a matrix of cofactors. We must first define what we mean by a tropical determinant. The standard definition for a tropical determinant of a $k \times k$ matrix M is

$$\operatorname{trop.det}(M) = \max_{\sigma \in S_k}(m_{1\sigma(1)} + \ldots + m_{k\sigma(k)}).$$

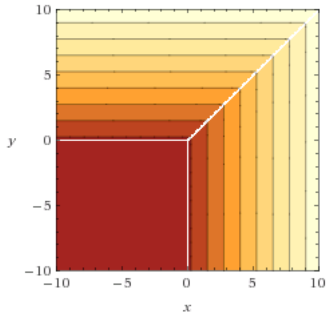
For a 2×2 matrix M this is simply

$$\operatorname{trop.det}(M) = \max(m_{11} + m_{22}, m_{12} + m_{21}).$$

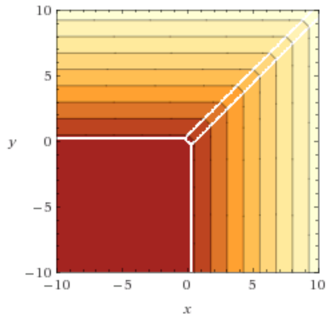
Using this, we can construct the tropical matrix of minors for S:

$$[\operatorname{trop.det}(S_{ij})]_{i,j} = \begin{bmatrix} \max(c+f, e+e) & \max(b+f, e+d) & \max(b+e, c+d) \\ \max(b+f, e+d) & \max(a+f, d+d) & \max(a+e, b+d) \\ \max(b+e, c+d) & \max(a+e, b+d) & \max(a+c, b+b) \end{bmatrix}.$$

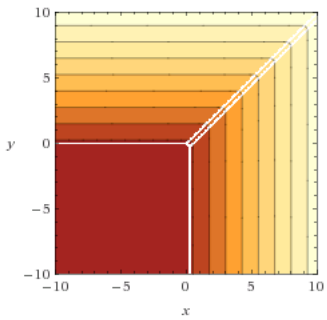
This tropical matrix of minors is our first proposed candidate for the dual matrix to S. Listed below are the representative symmetric matrices for four of the twenty types of tropical conics, along with the proposed dual matrices and the graphs of the corresponding double-max loci.



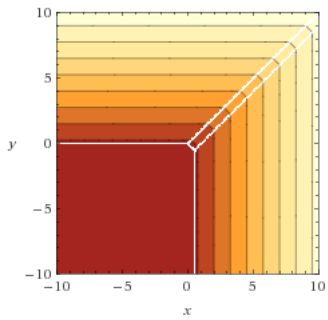
$$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \check{A}_1 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$



$$B_1 = \begin{bmatrix} 0 & 1 & 3/4 \\ 1 & 0 & 3/4 \\ 3/4 & 3/4 & 1 \end{bmatrix} \qquad \check{B}_1 = \begin{bmatrix} 3/2 & 2 & 7/4 \\ 2 & 3/2 & 7/4 \\ 7/4 & 7/4 & 2 \end{bmatrix}$$



$$C_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3/4 \\ 1 & 3/4 & 1 \end{bmatrix} \qquad \check{C}_1 = \begin{bmatrix} 3/2 & 2 & 7/4 \\ 2 & 2 & 2 \\ 7/4 & 2 & 2 \end{bmatrix}$$



$$D_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1/2 \\ 1 & 1/2 & 1 \end{bmatrix} \qquad \check{D}_1 = \begin{bmatrix} 1 & 2 & 3/2 \\ 2 & 2 & 2 \\ 3/2 & 2 & 2 \end{bmatrix}$$

Immediate concern arises from the graphs of these double-max loci. Matrices B_1, C_1, and D_1 represent tropical projective plane conics with different combinatorial properties, yet the double-max loci of the proposed duals all represent conics with similar combinatorial properties. Consequently, in creating a matrix of minors, as opposed to a matrix of cofactors, we appear to have lost important information.

To recover this information, we propose the consideration of matrix-pair duals for tropical conics. Simulating the classical matrix of cofactors and temporarily allowing for "tropical subtraction," we form the tropical cofactor matrix for S by

$$\begin{bmatrix} c \odot f \ominus e \odot e & \ominus(b \odot f \ominus e \odot d) & b \odot e \ominus c \odot d \\ \ominus(b \odot f \ominus e \odot d) & a \odot f \ominus d \odot d & \ominus(a \odot e \ominus b \odot d) \\ b \odot e \ominus c \odot d & \ominus(a \odot e \ominus b \odot d) & a \odot c \ominus b \odot b \end{bmatrix}$$

Note that the use of tropical negative here is purely formal and is merely being used to guide our search. That said, simplifying a bit we get

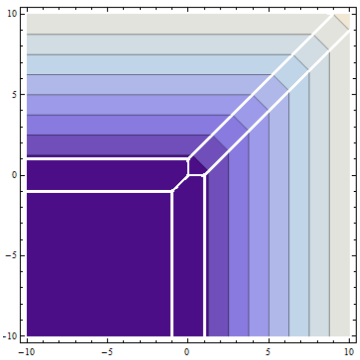
$$\begin{bmatrix} c\odot f\ominus e\odot e & e\odot d\ominus b\odot f & b\odot e\ominus c\odot d \\ e\odot d\ominus b\odot f & a\odot f\ominus d\odot d & b\odot d\ominus a\odot e \\ b\odot e\ominus c\odot d & b\odot d\ominus a\odot e & a\odot c\ominus b\odot b \end{bmatrix}$$

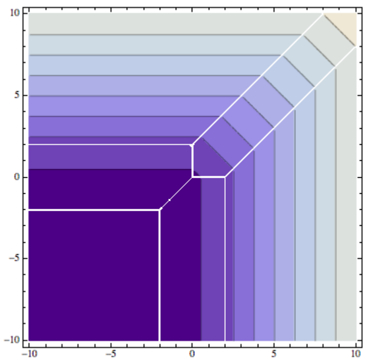
To rectify our use of tropical subtraction, we split this dual matrix into a pair of matrices. Define the positive and negative dual matrix, respectively, by

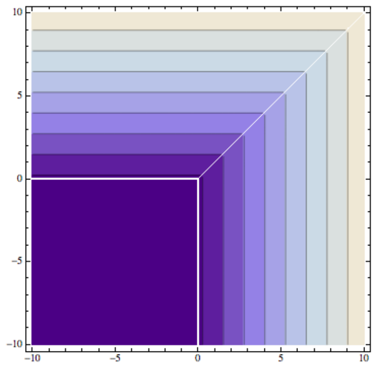
$$\check{S}^{+} = \begin{bmatrix} c\odot f & e\odot d & b\odot e \\ e\odot d & a\odot f & b\odot d \\ b\odot e & b\odot d & a\odot c \end{bmatrix} \qquad \check{S}^{-} = \begin{bmatrix} e\odot e & b\odot f & c\odot d \\ b\odot f & d\odot d & a\odot e \\ c\odot d & a\odot e & b\odot b \end{bmatrix}$$

Using this, we compute the newly proposed matrix-pair dual for each of the twenty types of tropical conics.

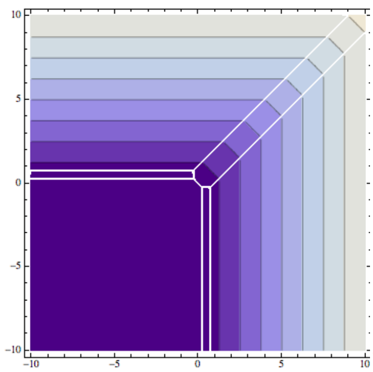
$$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \check{A}_1^{+} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \qquad \check{A}_1^{-} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

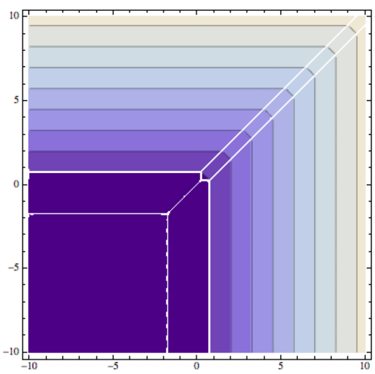


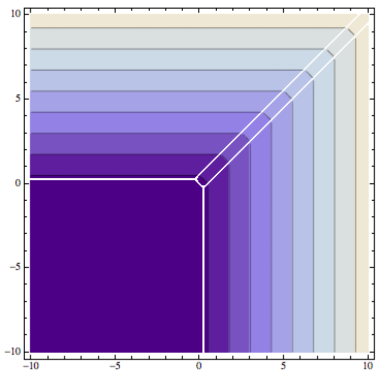




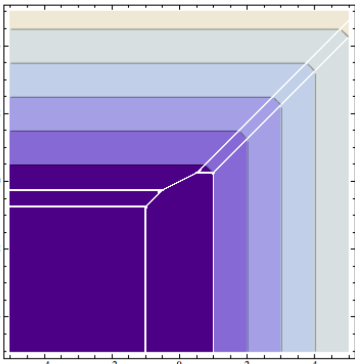
$$B_1 = \begin{bmatrix} 0 & 1 & 3/4 \\ 1 & 0 & 3/4 \\ 3/4 & 3/4 & 1 \end{bmatrix} \qquad \check{B}_1^{+} = \begin{bmatrix} 1 & 3/2 & 7/4 \\ 3/2 & 1 & 7/4 \\ 7/4 & 7/4 & 0 \end{bmatrix} \qquad \check{B}_1^{-} = \begin{bmatrix} 3/2 & 2 & 3/4 \\ 2 & 3/2 & 3/4 \\ 3/4 & 3/4 & 2 \end{bmatrix}$$

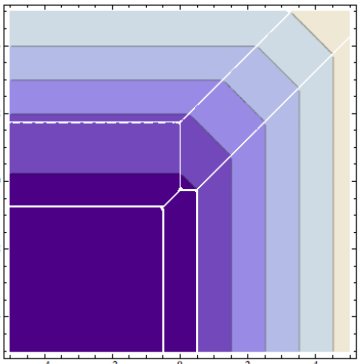


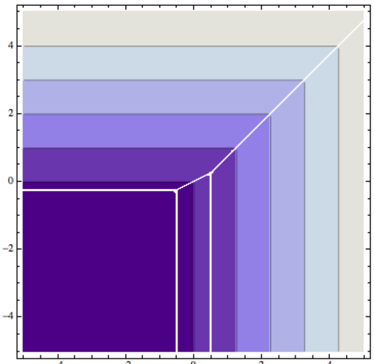




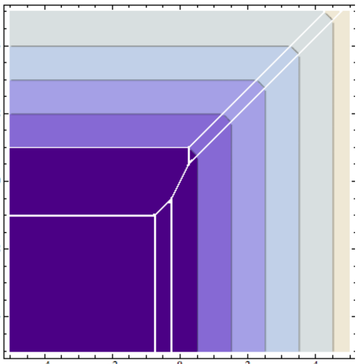
$$B_2 = \begin{bmatrix} 0 & 3/4 & 1 \\ 3/4 & 1 & 3/4 \\ 1 & 3/4 & 0 \end{bmatrix} \qquad \check{B}_2^{+} = \begin{bmatrix} 1 & 7/4 & 3/2 \\ 7/4 & 0 & 7/4 \\ 3/2 & 7/4 & 1 \end{bmatrix} \qquad \check{B}_2^{-} = \begin{bmatrix} 3/2 & 3/4 & 2 \\ 3/4 & 2 & 3/4 \\ 2 & 3/4 & 3/2 \end{bmatrix}$$

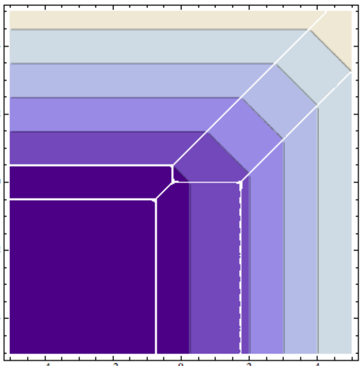


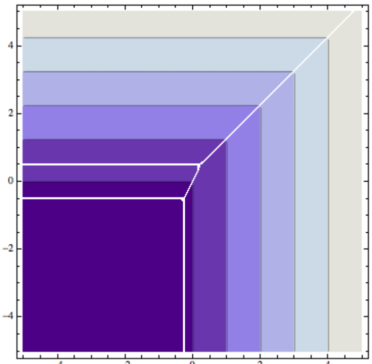




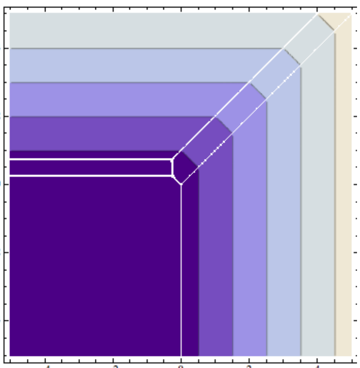
$$B_3 = \begin{bmatrix} 1 & 3/4 & 3/4 \\ 3/4 & 0 & 1 \\ 3/4 & 1 & 0 \end{bmatrix} \qquad \check{B}_3^{+} = \begin{bmatrix} 0 & 7/4 & 7/4 \\ 7/4 & 1 & 3/2 \\ 7/4 & 3/2 & 1 \end{bmatrix} \qquad \check{B}_3^{-} = \begin{bmatrix} 2 & 3/4 & 3/4 \\ 3/4 & 3/2 & 2 \\ 3/4 & 2 & 3/2 \end{bmatrix}$$

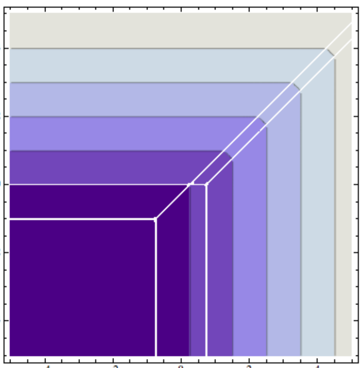


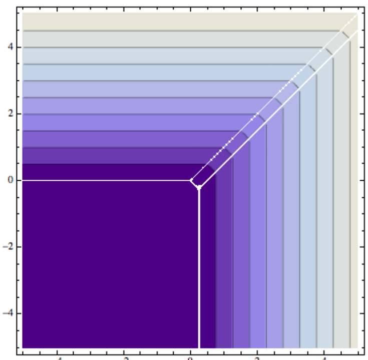




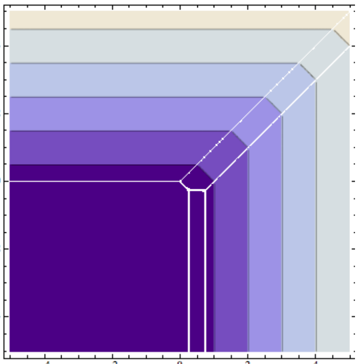
$$C_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3/4 \\ 1 & 3/4 & 1 \end{bmatrix} \qquad \check{C}_1^{+} = \begin{bmatrix} 1 & 7/4 & 7/4 \\ 7/4 & 2 & 2 \\ 7/4 & 2 & 1 \end{bmatrix} \qquad \check{C}_1^{-} = \begin{bmatrix} 3/2 & 2 & 1 \\ 2 & 2 & 7/4 \\ 1 & 7/4 & 2 \end{bmatrix}$$



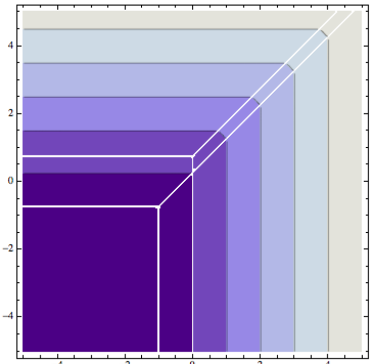




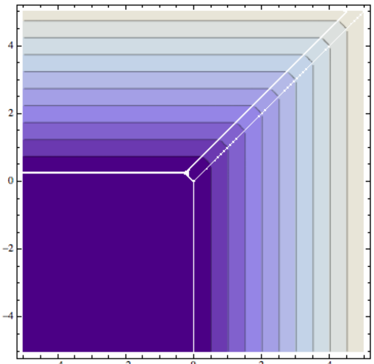
$$C_2 = \begin{bmatrix} 0 & 1 & {}^{3}/_{4} \\ 1 & 1 & 1 \\ {}^{3}/_{4} & 1 & 1 \end{bmatrix}$$



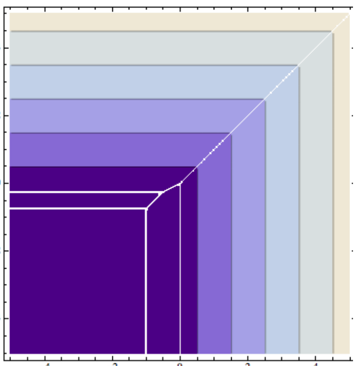
$$\check{C}_2^{+} = \begin{bmatrix} 2 & {}^{7}/_{4} & 2 \\ {}^{7}/_{4} & 1 & {}^{7}/_{4} \\ 2 & {}^{7}/_{4} & 1 \end{bmatrix}$$



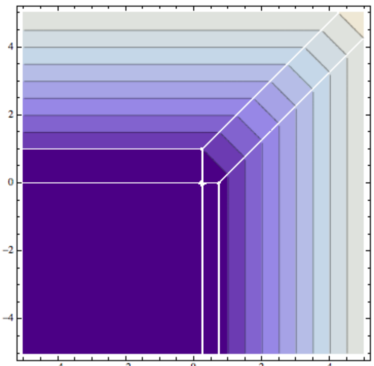
$$\check{C}_2^{-} = \begin{bmatrix} 2 & 2 & {}^{7}/_{4} \\ 2 & {}^{3}/_{2} & 1 \\ {}^{7}/_{4} & 1 & 2 \end{bmatrix}$$



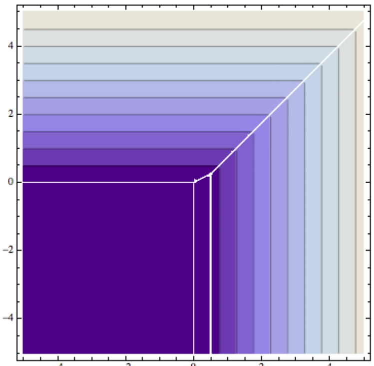
$$C_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & {}^{3}/_{4} \\ 1 & {}^{3}/_{4} & 0 \end{bmatrix}$$



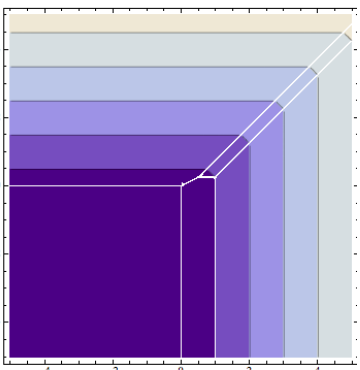
$$\check{C}_3^{+} = \begin{bmatrix} 1 & {}^{7}/_{4} & {}^{7}/_{4} \\ {}^{7}/_{4} & 1 & 2 \\ {}^{7}/_{4} & 2 & 2 \end{bmatrix}$$



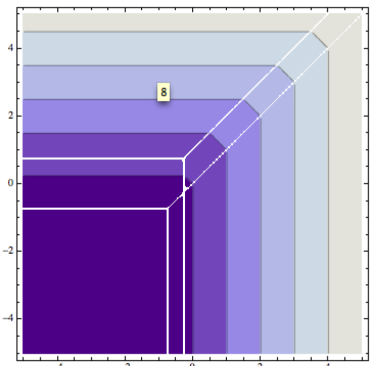
$$\check{C}_3^{-} = \begin{bmatrix} {}^{3}/_{2} & 1 & 2 \\ 1 & 2 & {}^{7}/_{4} \\ 2 & {}^{7}/_{4} & 2 \end{bmatrix}$$



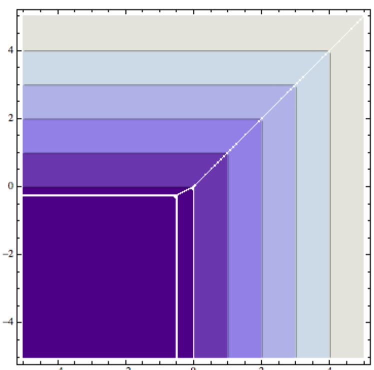
$$C_4 = \begin{bmatrix} 0 & {}^{3}/_{4} & 1 \\ {}^{3}/_{4} & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



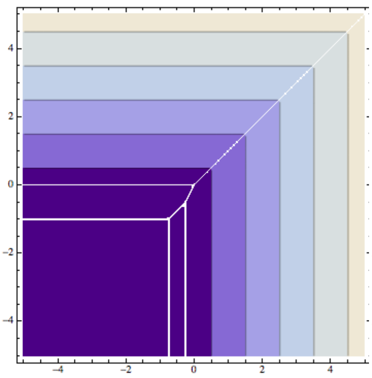
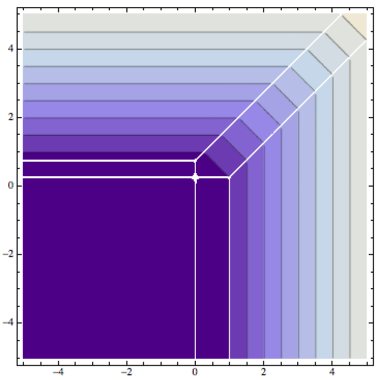
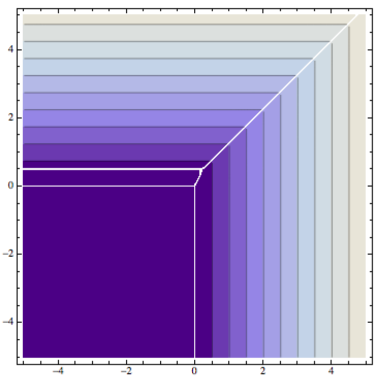
$$\check{C}_4^{+} = \begin{bmatrix} 2 & 2 & {}^{7}/_{4} \\ 2 & 1 & {}^{7}/_{4} \\ {}^{7}/_{4} & {}^{7}/_{4} & 1 \end{bmatrix}$$



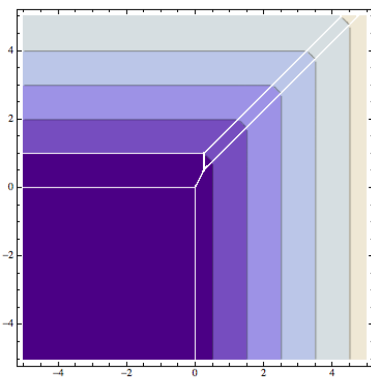
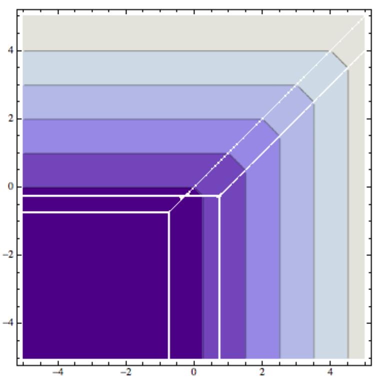
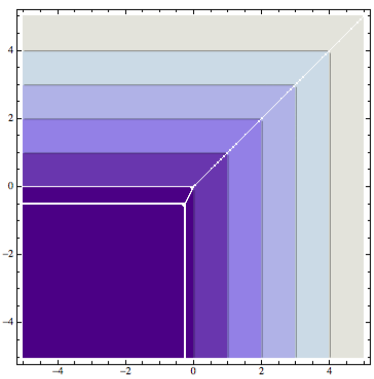
$$\check{C}_4^{-} = \begin{bmatrix} 2 & {}^{7}/_{4} & 2 \\ {}^{7}/_{4} & 2 & 1 \\ 2 & 1 & {}^{3}/_{2} \end{bmatrix}$$



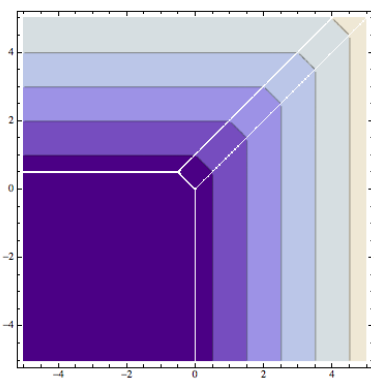
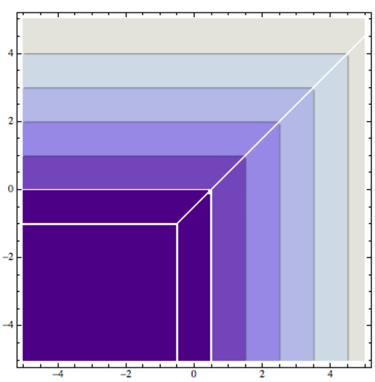
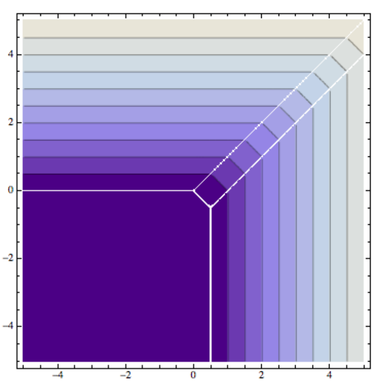
$$C_5 = \begin{bmatrix} 1 & 1 & 3/4 \\ 1 & 1 & 1 \\ 3/4 & 1 & 0 \end{bmatrix} \qquad \check{C}_5^+ = \begin{bmatrix} 1 & 7/4 & 2 \\ 7/4 & 1 & 7/4 \\ 2 & 7/4 & 2 \end{bmatrix} \qquad \check{C}_5^- = \begin{bmatrix} 2 & 1 & 7/4 \\ 1 & 3/2 & 2 \\ 7/4 & 2 & 2 \end{bmatrix}$$

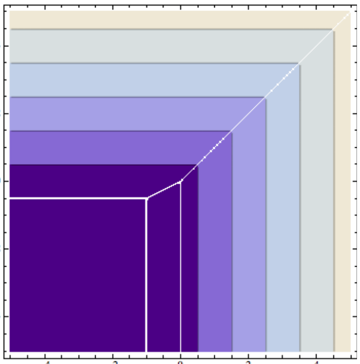
$$C_6 = \begin{bmatrix} 1 & 3/4 & 1 \\ 3/4 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \check{C}_6^+ = \begin{bmatrix} 1 & 2 & 7/4 \\ 2 & 2 & 7/4 \\ 7/4 & 7/4 & 1 \end{bmatrix} \qquad \check{C}_6^- = \begin{bmatrix} 2 & 7/4 & 1 \\ 7/4 & 2 & 2 \\ 1 & 2 & 3/2 \end{bmatrix}$$

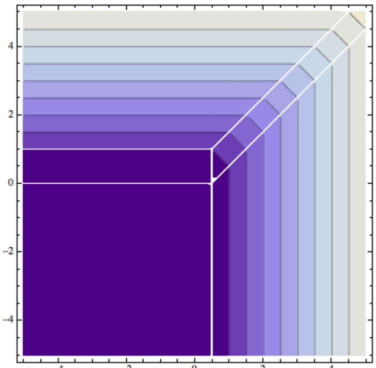
$$D_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1/2 \\ 1 & 1/2 & 1 \end{bmatrix} \qquad \check{D}_1^+ = \begin{bmatrix} 1 & 3/2 & 3/2 \\ 3/2 & 2 & 2 \\ 3/2 & 2 & 1 \end{bmatrix} \qquad \check{D}_1^- = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3/2 \\ 1 & 3/2 & 2 \end{bmatrix}$$

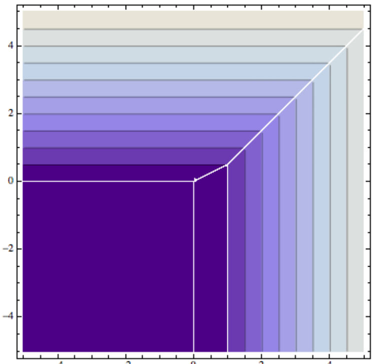
$$D_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & {}^1\!/_2 \\ 1 & {}^1\!/_2 & 0 \end{bmatrix}$$



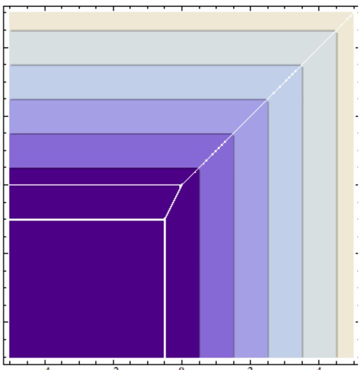
$$\check{D}_2^+ = \begin{bmatrix} 1 & {}^3\!/_2 & {}^3\!/_2 \\ {}^3\!/_2 & 1 & 2 \\ {}^3\!/_2 & 2 & 2 \end{bmatrix}$$



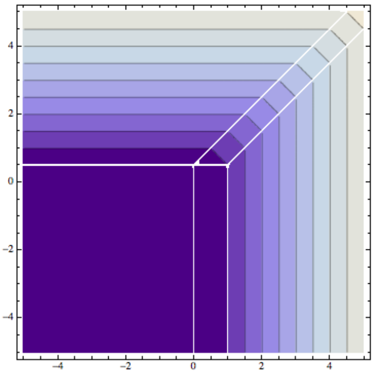
$$\check{D}_2^- = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & {}^3\!/_2 \\ 2 & {}^3\!/_2 & 2 \end{bmatrix}$$



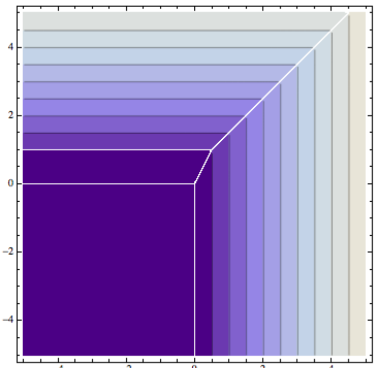
$$D_3 = \begin{bmatrix} 1 & 1 & {}^1\!/_2 \\ 1 & 1 & 1 \\ {}^1\!/_2 & 1 & 0 \end{bmatrix}$$



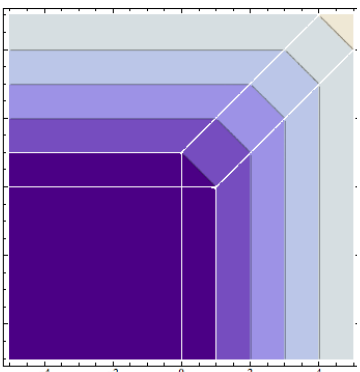
$$\check{D}_3^+ = \begin{bmatrix} 1 & {}^3\!/_2 & 2 \\ {}^3\!/_2 & 1 & {}^3\!/_2 \\ 2 & {}^3\!/_2 & 2 \end{bmatrix}$$



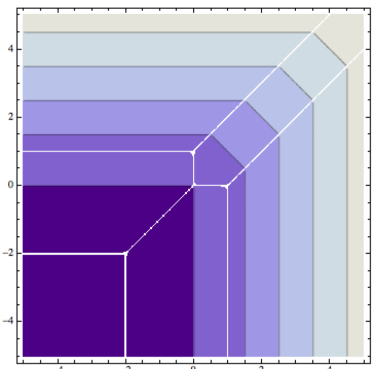
$$\check{D}_3^- = \begin{bmatrix} 2 & 1 & {}^3\!/_2 \\ 1 & 1 & 2 \\ {}^3\!/_2 & 2 & 2 \end{bmatrix}$$



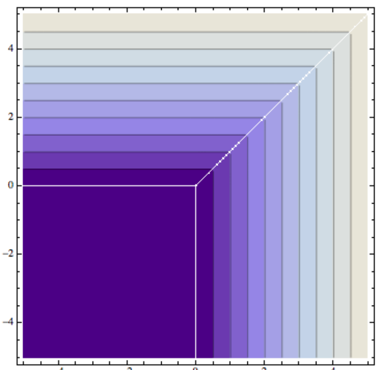
$$E_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



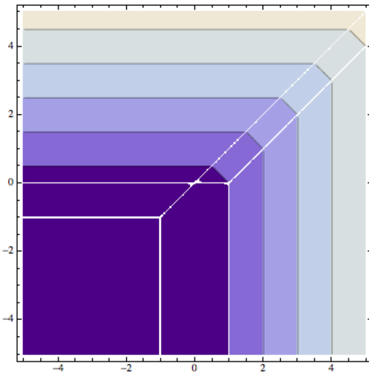
$$\check{E}_1^+ = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

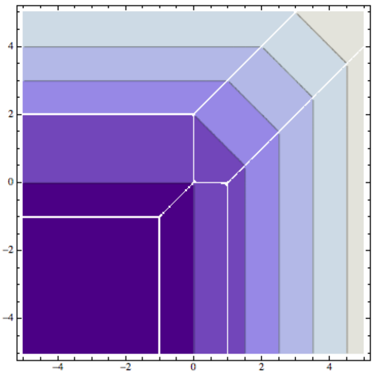


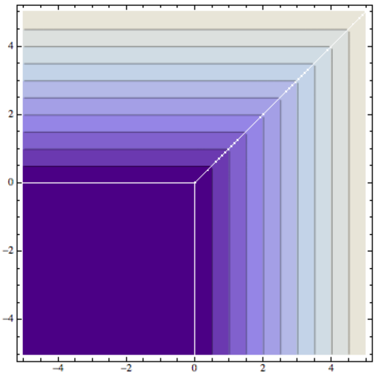
$$\check{E}_1^- = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$



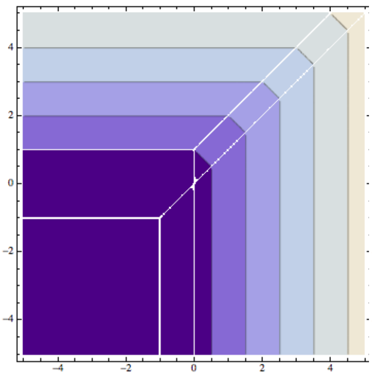
$$E_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \check{E}_2^{+} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 1 \end{bmatrix} \qquad \check{E}_2^{-} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

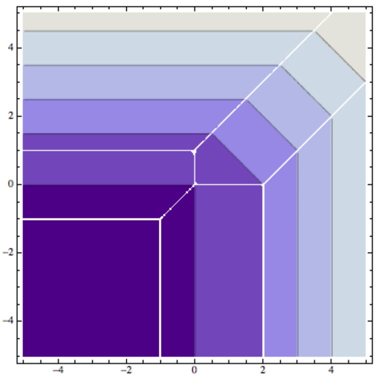


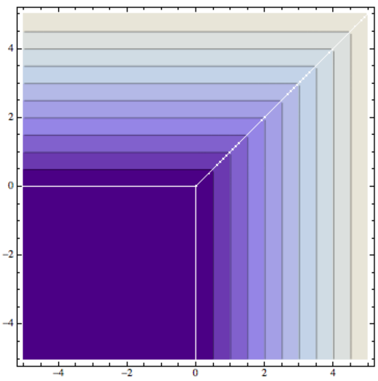




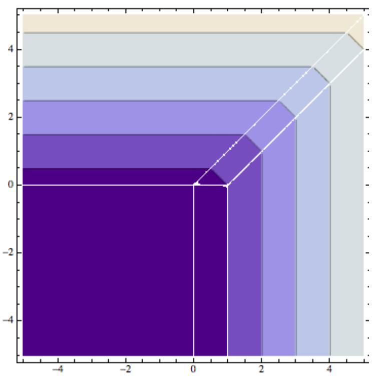
$$E_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \check{E}_3^{+} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \qquad \check{E}_3^{-} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

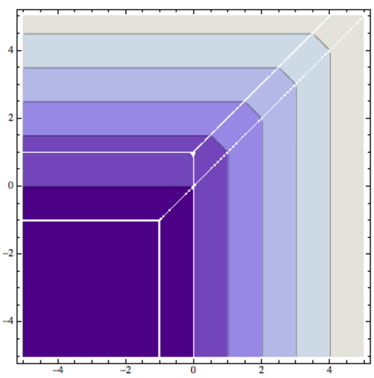


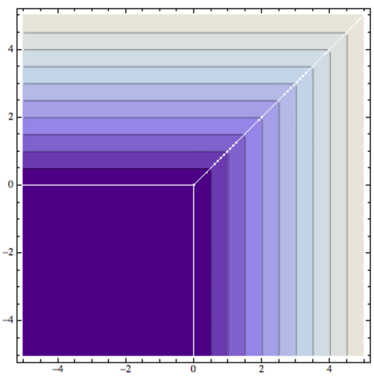




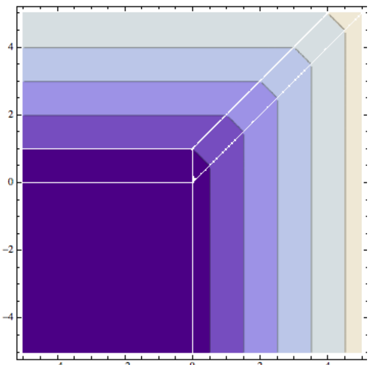
$$F_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \check{F}_1^{+} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \qquad \check{F}_1^{-} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

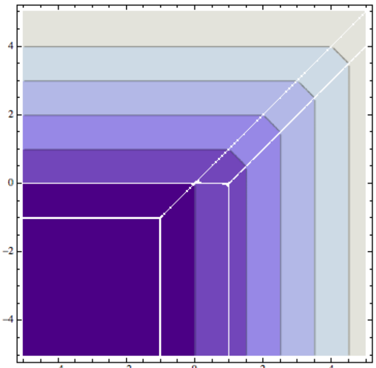


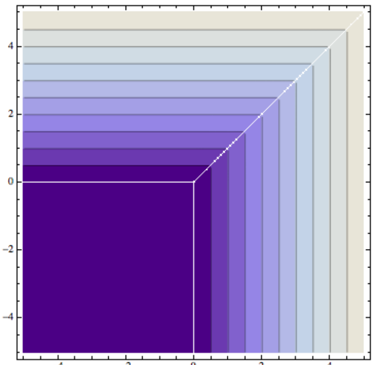




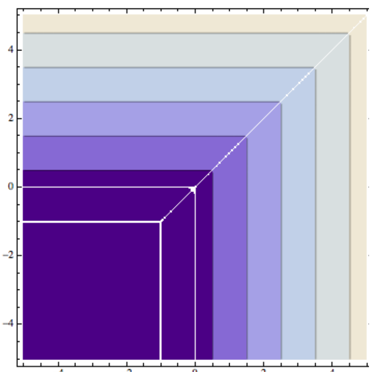
$$F_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \check{F}_2^+ = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix} \qquad \check{F}_2^- = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

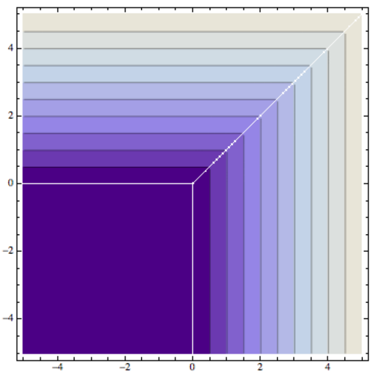


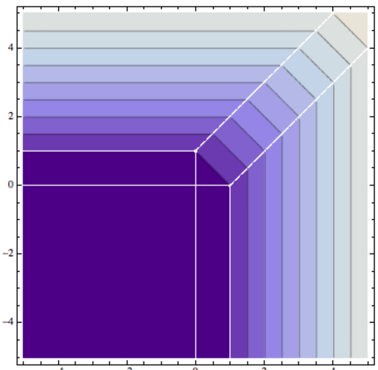




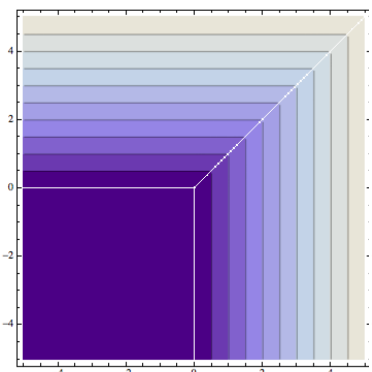
$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \check{F}_3^+ = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} \qquad \check{F}_3^- = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

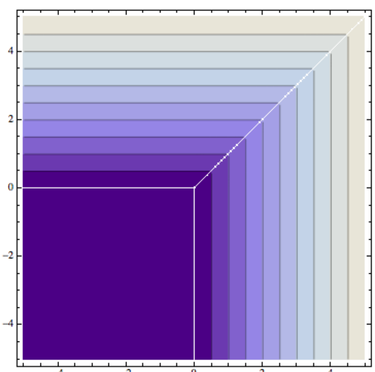


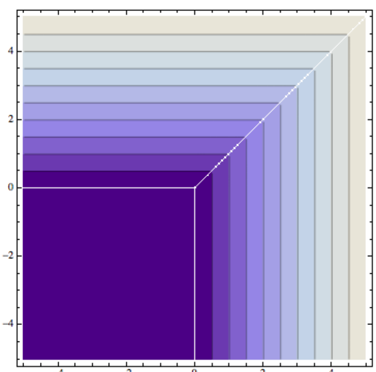




$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \check{G}_1^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \check{G}_1^- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$







We organize this information in the following table by identifying each of the matrices in the proposed dual matrix-pair as one of the twenty representative types of tropical projective plane conics.

S	A_1	B_1	B_2	B_3	C_1	C_2	C_3	C_4	C_5	C_6	D_1	D_2	D_3	E_1	E_2	E_3	F_1	F_2	F_3	G_1
$\check{S}^+$	A_1	A_1	A_1	A_1	E_2	E_3	E_1	E_3	E_1	E_2	F_3	F_2	F_1	A_1	A_1	A_1	E_3	E_2	E_1	G_1
$\check{S}^-$	G_1	D_1	D_2	D_3	D_1	D_1	D_2	D_2	D_3	D_3	D_1	D_2	D_3	G_1	G_1	G_1	G_1	G_1	G_1	G_1

Examining these associations, there appears to be the first hint at some form of tropical duality. However, there is also clearly some defect in this approach, as the duality appears to break down in the latter half of the list. So while this construction is promising, it is not clear at this point what should be done to correct the defects.

3.3 Duality via the graph equations

A third and final option for a tropical dual map is to consider tropicalizing the equations defining closure of the graph of the classical dual map, i.e., the blowup of $\mathbf{P}^5$ along the Veronese surface. Unfortunately, this approach also appears to have serious issues. Indeed, for many of the twenty combinatorial types of tropical conics, the tropicalized graph equations force all of the coefficients in the proposed dual conic to equal $-\infty$. It is likely that the eight syzygy equations are causing problems, since their computation classically involves a great deal of subtraction. Terms that might cancel in the classical case would not cancel if the analogous computations were performed in the tropical setting. So it seems that direct tropicalization of the graph equations is also not the correct approach.

4 Areas for Future Research

In this project we explored three natural possibilities for a tropical dual map. Unfortunately, in each approach we encountered serious issues. The lack of an established notion of tropical tangency complicated the approach via tangent lines, while the lack of tropical subtraction complicated the dual symmetric matrices and sygyzy equations. Fortunately, though, there are still additional possibilities to consider.

One possible approach is to consider a more nuanced tropicalization of the twenty-three graph equations. Instead of directly tropicalizing the sygyzy equations from [BKT08], one could attempt to derive tropical analogues of those equations from the first fifteen (tropicalized) equations. This computation likely will be tedious to do by hand, so access to a mathematical program that can solve systems of equations (involving maxima) is highly recommended.

A second possible approach is the consideration of congruences on the tropical semiring. Congruences, which are certain generalizations of ideals on semirings, have recently been explored as interesting tropical objects in their own rights. Whereas tropical curves as defined here are meant to represent the images of classical complex curves under a tropicalization map, congruences (and their corresponding congruence varieties) are meant to directly generalize the classical construction of algebraic varieties in this new tropical setting. In particular, congruences and congruence varieties are the most straightforward way to modify classical constructions that involve substraction in this new setting in which subtraction no longer exists. It is therefore reasonable that the correct description of the tropicalization of the closure of the graph of the classical dual map might be via congruences and congruence varieties, as opposed to the corner-loci considered here. For more information on congruences and congruence varieties, see [BE14].

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