CLASSICAL AND TROPICAL ALGEBRAIC GEOMETRY

NICKY EVANS

1. Introduction

This paper is an introduction to the worlds of algebraic and tropical geometry. Both subjects give connections between the seemingly separate subjects of algebra and geometry. On the algebra side, we are mainly concerned with polynomials and subsets of polynomial rings called "ideals." On the geometry side, we are concerned with geometric objects like curves and surfaces. The two sides are linked together by considering zero sets of polynomials.

In the classical picture, everything is taking place over a field. Usually the field is algebraically closed, like the complex numbers. In the tropical picture, however, we are working with the so-called "tropical numbers." These numbers act a lot like the real numbers, only with different operations (replacing the familiar operations of addition and multiplication). Tropical geometry originally arose through the study of logarithmic images of complex curves. These images, called "amoebas," could largely be understood through certain piece-wise linear information encoded in their "skeletons." Tropical geometry is a very piece-wise linear geometry, which makes many algebro-geometric problems more malleable in the tropical setting. Tools like combinatorics and graph theory, for instance, can be used to attack previously intractable problems.

2. Classical Algebraic Geometry

This section discusses the classical connection between algebra and geometry. On the algebra side, we have polynomials and collections of polynomials called *ideals*. On the geometry side, we have subsets of space where polynomials, or collections of polynomials, vanish. These subsets are called *varieties*. This section will explore the close connection that exists between ideals and varieties. In order to understand varieties, we first study rings and polynomials over a field. A field is a set where one can define addition, subtraction, multiplication and division. For example, the real numbers $\mathbf{R}$ and the complex numbers $\mathbf{C}$ are fields, while the integers $\mathbf{Z}$ is not because division fails (3 and 2 are integers, however their quotient $3/2$ is not). Our first step to understanding polynomials is to approach rings and exploit their properties.

Date: June 12, 2015.

2.1. **Algebra.**

In the algebra sections, we will generally be working with polynomials, and the set of polynomials has an algebraic structure known as a *ring*.

Definition 2.1. A **ring** is a set R together with two binary operations, called addition and multiplication and denoted $+$ and $\cdot$, respectively, satisfying the following three axioms:

i. $(R, +)$ is an abelian group: there is an additive identity (denoted 0), every element has an additive inverse, and addition is both commutative and associative;
ii. $(R, \cdot)$ is a monoid: it contains a multiplicative identity (denoted 1) and multiplication is associative; and
iii. Multiplication is distributive under addition, i.e., $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot a + c \cdot a$ for every $a, b, c \in R$.

Example 2.2. A familiar example of a ring is the integers $\mathbf{Z}$. Addition is commutative and associative, since $a + b = b + a$ and $(a + b) + c = a + (b + c)$. The ring $\mathbf{Z}$ contains an additive inverse $-n$ for any integer n, since $n + (-n) = 0$, and the ring contains the additive identity 0, since for an integer $n, n + (0) = n$. The second condition also holds. There exists a multiplicative identity 1 for any integer n, since $1 \cdot n = n$, and multiplication is associative, since $a \cdot (b \cdot c) = a \cdot b \cdot c = (a \cdot b) \cdot c$ for $a, b, c \in \mathbf{Z}$. Lastly, multiplication between integers is distributive, since $a \cdot (b + c) = a \cdot b + a \cdot c = (b + c) \cdot a$ for $a, b, c \in \mathbf{Z}$.

A field is just a commutative ring in which every nonzero element has a multiplicative inverse, so notice that the ring of positive integers is not a field because their exists no additive inverse.

Example 2.3. Another example is the real numbers $\mathbf{R}$. This ring can be checked in a similar manner as the previous example.

Although rings can be composed of various objects from mathematics, we will mainly be concerned with rings of polynomials.

Definition 2.4. A **monomial** in $x_1, x_2, \ldots, x_n$ is a product of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ where all exponents are nonnegative integers. The total degree of a monomial is the sum $\alpha_1 + \alpha_2 + \cdots + \alpha_n$, denoted $\mid \alpha \mid$.

Definition 2.5. A **polynomial** f in $x_1, \ldots, x_n$ with coefficients in a ring R is a finite linear combination of monomials, denoted $f(x) = \sum_\alpha a_\alpha x^\alpha$, where $a_\alpha \in R$, all but finitely many of which are zero. The set of all polynomials in $x_1, \ldots, x_n$ with coefficients in R is denoted $R[x_1, \ldots, x_n]$.

Example 2.6. An example of a polynomial lying in $\mathbf{Q}[x, y]$ is $f(x, y) = 3x^3 - \frac{1}{2}xy^2 + y^3$.

Example 2.7. A polynomial in $\mathbf{Z}[w, x, y, z]$ is $f(w, x, y, z) = 4x + y^3z^4 - 7wxy$.

Every polynomial defines a function $f : \mathbf{R}^n \to \mathbf{R}$, and this leads into the Geometry section in which we see how polynomials can be manipulated.

2.2. **Geometry.**

We next make a connection between our algebraic objects (polynomials) and some geometric objects. We first define the space in which our geometric objects will live.

Definition 2.8. Given a field k along with a positive integer n, the n-dimensional **affine space** over k is k^n=$\{(a_1...a_n) \mid a_1...a_n \in k\}$

Example 2.9. The affine space k^1 is the affine line, or more notably, if the field k is $\mathbf{R}$, then $\mathbf{R}^1$ is the real line. Similarly, the affine space k^2 is the affine plane and $\mathbf{R}^2$ is the real plane we regularly use.

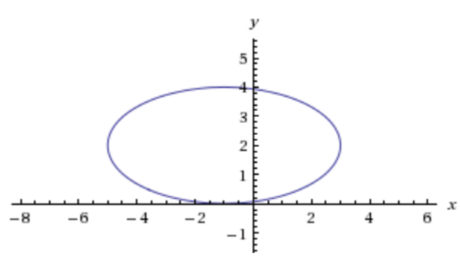
In order to link algebra and geometry, we must be able to view a polynomial as a function. Observe that any polynomial $f(x) = \sum_\alpha x^\alpha \in k[x_1...x_n]$ gives a function $f : k^n \to k$, defined as follows: given $(a_1...a_n) \in k^n$, replace every x_i by a_i in the expression for f. Since all the coefficients also lie in k, this operation gives an element $f(a_1...a_n) \in k$. We can now move between algebra and geometry in affine space with varieties.

Definition 2.10. Let k be a field, and let $f_1, \dots, f_s$ be polynomials in $k[x_1, \dots, x_n]$. The **affine variety** defined by $f_1, \dots, f_s$ is the set

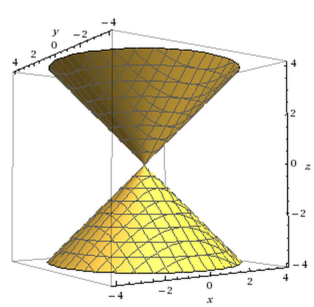
$$\mathbf{V}(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in k^n \mid f_i(a_1, \dots, a_n) = 0, 1 \leqslant i \leqslant s\} \subseteq k^n.$$

Example 2.11.

a. Consider the variety $\mathbf{V}(x^2 + 4y^2 + 2x - 16y + 1) \subseteq \mathbf{R}^2$. By definition, this variety is the curve defined by the equation $x^2+4y^2+2x-16y+1 = 0$. One can check that this equation is equivalent to the equation $(x+1)^2+4(y-2)^2 = 16$, or equivalently $\left(\frac{x+1}{4}\right)^2 + \left(\frac{y-2}{2}\right)^2 = 1$. This is the equation of an ellipse centered at $(-1, 2)$, as shown:



b. Consider the variety $\mathbf{V}(z^2 - x^2 - y^2) \subseteq \mathbf{R}^3$. This is the surface defined by the equation $z^2 - x^2 - y^2 = 0$, or equivalently $z^2 = x^2 + y^2$. This is the familiar equation of a circular cone:



Example 2.12. Let $\mathbf{H} = \{(a,b) \in \mathbf{R}^2 \mid b > 0\}$ be the upper half plane. We claim $\mathbf{H}$ is not an affine variety. Suppose $\mathbf{H}$ is an affine variety, say $\mathbf{H} = \mathbf{V}(f_1, \ldots, f_s)$. Then each polynomial f_i must vanish on $\mathbf{H}$. However, any polynomial that vanishes on the upper half-plane must vanish on the entire plane, and hence be the zero polynomial. But then $\mathbf{V}(f_1, \ldots, f_s) = \mathbf{V}(0) = \mathbf{R}^2 \neq \mathbf{H}$. Therefore $\mathbf{H}$ is not an affine variety.

2.3. **The Algebra-Geometry Correspondence.**

We have seen some simple ways to travel back and forth between the algebra and the geometry through varieties, and we now establish a close correspondence between the two. To each subset $S \subseteq k^n$ we can associate the subset

$$\mathbf{I}(S) = \{f \in k[x_1, \ldots, x_n] \mid f(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in S\}.$$

This collection of polynomials has some special properties, namely, it is an *ideal.*

Definition 2.13. Let R be a commutative ring. A subset $I \subseteq R$ is an **ideal** if it satisfies the following three properties:

i. $0 \in I$.
ii. If $f, g \in I$, then $f + g \in I$.
iii. If $f \in I$ and $r \in R$, then $rf \in I$.

We will generally be concerned with ideals in a polynomial ring $k[x_1, ..., x_n]$, where k is a field. We will denote the ideal generated by polynomials $f_1, ..., f_n$ in $k[x_1, ..., x_n]$ as $\langle f_1, ..., f_n\rangle$. The ideal $\langle f_1, ..., f_n\rangle$ is the smallest ideal containing those elements, and it can also be described as the subset of all elements of the form $\sum_{i=}^{r} r_i f_i$, where $r_i \in R$.. Ideals have relatively simple interpretations in terms of polynomial equations. Given $f_1, ..., f_n \in k[x_1, ..., x_n]$, we can form the system of equations $f_1 = 0, ..., f_n = 0$. Polynomial properties can also be taken advantage of if we multiply our first equation f_1 by $h_1 \in k[x_1, ..., x_n]$, our second equation f_2 by $h_2 \in k[x_1, ..., x_n]$ and so forth. One consequence of our original system is the polynomial equation $h_1 f_1 + h_2 f_2 + ... + h_n f_n = 0$. The left-hand side is exactly an element of the ideal $\langle f_1, ..., f_n\rangle$. So, we can think of $\langle f_1, ..., f_n\rangle$ as encoding all polynomial consequences of the equations $f_1 = ... = f_n = 0$.

Example 2.14. Ideals can have many different sets of generators, as the following examples illustrate.

a. In the ring $\mathbf{Q}[x, y]$ we claim that $\langle x + y, x - y\rangle = \langle x, y\rangle$. To show this equality holds, we simply need to check that $\langle x + y, x - y\rangle \subseteq \langle x, y\rangle$ and $\langle x + y, x - y\rangle \supseteq \langle x, y\rangle$. We notice $x+y = 1\cdot x+1\cdot y \in \langle x, y\rangle$ and $x-y = 1\cdot x-1\cdot y \in \langle x, y\rangle$. Now for the reverse containment. We can see that by a multiple of 1/2 we get $x = 1/2(x + y) + 1/2(x - y) \in \langle x + y, x - y\rangle$ and $y = 1/2(x + y) - 1/2(x - y) \in \langle x + y, x - y\rangle$. Thus $\langle x + y, x - y\rangle = \langle x, y\rangle$.

b. In the ring $\mathbf{Q}[x, y]$, we claim that $\langle 2x^2 + 3y^2 - 11, x^2 - y^2 - 3\rangle = \langle x^2 - 4, y^2 - 1\rangle$. We can follow the same process as in the previous exercise. First notice that $2x^2 + 3y^2 - 11 = 2(x^2 - 4) + 3(y^2 - 11)$. This implies $2x^2 + 3y^2 - 11 \in \langle x^2 - 4, y^2 - 11\rangle$. Also, $x^2 - y^2 - 3 = 1(x^2 - 4) - (y^2 - 11)$. Then the reverse containment follows upon noting: $5(x^2-4) = (2x^2+3y^2-11)+3(x^2-y^2-3)$ and $5(y^2-1) = (2x^2+3y^2-11)-2(x^2-y^2-3)$. Thus, the equality of ideals holds.

It is straightforward to check that if $V \subset k^n$ is an affine variety, then the set $\mathbf{I}(V) \subset k[x_1, \ldots, x_n]$ is an ideal. Obviously, we call $\mathbf{I}(V)$ the ideal of V. Conversly, to each ideal I we can define the variety

$$\mathbf{V}(I) = \{(a_1, \ldots, a_n) \in k^n \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}.$$

We thus have our connection between ideals and varieties. It is not clear, however, to what extent these associations are inverses. For a more visual display, the process can be seen as follows: $f_1, \ldots, f_s \to \mathbf{V}(f_1, \ldots, f_s) \to \mathbf{I}(\mathbf{V}(f_1, \ldots, f_s))$. So, does $\mathbf{I}(\mathbf{V}(f_1, \ldots, f_s)) = \langle f_1, \ldots, f_s \rangle$? The closest immediate answer to this question is: If $f_1, ..., f_s \in k[x_1, \ldots, x_n]$, then $\langle f_1, \ldots, f_s \rangle \subset \mathbf{I}(\mathbf{V}(f_1, \ldots, f_s))$. We also have the following nice property, which tells us these associations are inclusion-reversing:

Theorem 2.15. [H77, Prop. 1.2, pg. 3] *Let V and W be affine varieties in k^n.*

i. *$V \subset W$ if and only if $\mathbf{I}(V) \supset \mathbf{I}(W)$*
ii. *$V = W$ if and only if $\mathbf{I}(V) = \mathbf{I}(W)$*

We have constructed a basic dictionary to be used when transitioning from algebra to geometry. Ideals in algebra and varieties in geometry allow new insights, as we have seen in some previous example, and by transforming problems back and forth, we can uncover ideas and solution previously hidden. Although this dictionary comes with many answers, it also comes with many questions, mainly:

- Ideal Description: can every ideal $I \subset k[x_1, \ldots, x_n]$ be written as $\langle f_1, \ldots, f_s \rangle$ for some $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$?
- Ideal Membership: if $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, is there an algorithm to decide whether a given $f \in k[x_1, \ldots, x_n]$ lies in $\langle f_1, \ldots, f_s \rangle$?
- Nullstellensatz: given $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, what is the exact relation between $\langle f_1, \ldots, f_s \rangle$ and $\mathbf{I}(\mathbf{V}(f_1, \ldots, f_s))$?

The last question will be answered in Chapter 3, but first we delve into the tropical setting.

3. Tropical Algebraic Geometry

3.1. **Tropical Algebra.**

The set of tropical numbers is similar to the set of real numbers, but the operations we use are different for the two sets. The new structure on tropical numbers is used to study geometric objects, called amoebas, and these objects become beneficial in their inherent ability to translate algebraic questions into various mathematical questions in other topics, such as graph theory or combinatorics. First, let's start with the algebraic structure of the tropical numbers.

Definition 3.1. The **tropical numbers** is the set $\mathbf{T} = \mathbf{R} \cup \{-\infty\}$ with the two operations, which we call **tropical addition** and **tropical multiplication**:

a. $x \oplus y = \max\{x, y\}$
b. $x \odot y = x + y$

Note that $\mathbf{T}$ possesses many of the properties of a ring. For instance, the element $-\infty$ is the additive identity, while the real number 0 acts as the multiplicative identity. Indeed, $x \oplus -\infty = \max\{x, -\infty\} = x$ and $x \odot 0 = x + 0 = x$. Tropical addition and tropical multiplication are both commutative and associative. The main new feature, however, is that tropical addition is idempotent, i.e., $x \oplus x = x$ for every $x \in \mathbf{T}$. As a result, almost no tropical number has an additive inverse. For a tropical number x to have an additive inverse, there would need to be some $y \in \mathbf{T}$ such that $x \oplus y = -\infty$. This is only possible when $x = y = -\infty$. Therefore, tropical subtraction does not exist. This is the only property that prevents $\mathbf{T}$ from being a ring, and so $\mathbf{T}$ is dubbed a *semiring*. The additive identity is $-\infty$, denoted $0_{\mathbf{T}}$; the multiplicative identity is 0, denoted $1_{\mathbf{T}}$.

Example 3.2.

a. Consider the number $1_{\mathbf{T}} \oplus 5$ in $\mathbf{T}$. In $\mathbf{R}$, this translates to $\max\{0, 5\}$, which is 5.
b. Consider the tropical function $f(x) = x \oplus x^{\odot 2}$. Working in $\mathbf{R}$, this translates to $\max\{x, 2x\}$. This is a piece-wise linear function:

$$f(x) = \begin{cases} x & : x < 0 \\ 2x & : x \geq 0 \end{cases}$$

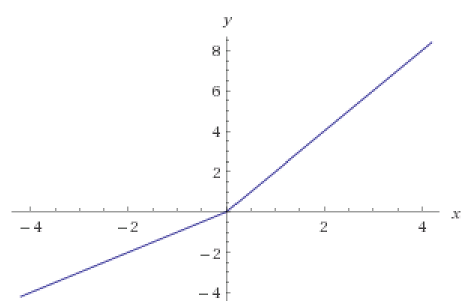


FIGURE 1. The graph of f.

c. Consider the equation $2 \odot x^{\odot 4} = 0_{\mathbf{T}}$. This equation translates to $2 + 4x = -\infty$, whose only solution is $x = -\infty = 0_{\mathbf{T}}$.

Similarly to classical polynomials, tropical polynomials naturally induce functions $f : \mathbf{T}^n \to \mathbf{T}$. We can now get to the interesting part of actually using this semiring to make graphs that deliver new insights to our original algebraic terms.

3.2. **Tropical Geometry.**

A recurring problem with $\mathbf{T}$ is the fact that it is a semiring- there is not subtraction! This changes our scope of thinking slightly. Typically, affine varieties are studied by observing where a function equals zero in order to study the zero loci of classical polynomials, so instead of looking at where points of a function vanish, we look at points where two functions are equal, i.e., $\mathbf{V}((f, g)) = \{a \in \mathbf{T}^n \mid f(a) = g(a)\}$, for tropical polynomials $f, g \in \mathbf{T}[x]$.

The following examples nicely show to find a set of solutions that will be required to display skeleton-like graphs named amoebas.

Example 3.3. Compute the following:

a. What is $\mathbf{V}((1_{\mathbf{T}} \oplus x, x))$? This asks the question, "When does $1_{\mathbf{T}} \oplus x = x$?" When $x < 0$, the left-hand side is $\max\{0, x\} = 0$, while the right-hand side is $x < 0$, so the two sides cannot be equal. When $x \geq 0$, the left-hand side is $\max\{0, x\} = x$ and the right-hand side is x, which are always equal. Therefore, the answer is $\{a \in \mathbf{T} \mid a \geq 0\}$
b. What is $\mathbf{V}((1_{\mathbf{T}} \oplus x^{\odot 2}, x))$? These two functions are equal at a single point $\{a \in \mathbf{T} \mid a = 0\}$.
c. What is $\mathbf{V}((1_{\mathbf{T}} \oplus (2 \odot x) \oplus x^{\odot 2}, 1_{\mathbf{T}} \oplus x^{\odot 2}))$? Let's rewrite this one in real terms: $\max\{0, 2+x, 2x\}$ and $\max\{0, 2x\}$. This makes the variety more clear. These two functions are equal when x is less than zero, or x is greater than or equal to 2, i.e. $\{a \in \mathbf{T} \mid a < 0 \text{ or } a \geq 2\}$.

Example 3.4. Here are some two-dimensional examples:

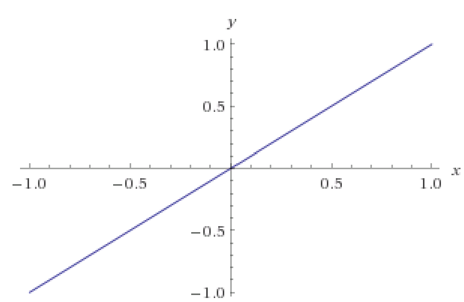
a. Wha is $\mathbf{V}((1_{\mathbf{T}}, x), (0_{\mathbf{T}}, y))$? This is the question,"Where is $x = 1_{\mathbf{T}}$ and $y = 0_{\mathbf{T}}$?" The question posed gives the answer: Exactly at the point $(1_{\mathbf{T}}, 0_{\mathbf{T}})$. This point lies very far down on the y-axis!
b. What is $\mathbf{V}((1_{\mathbf{T}} \oplus x, 1_{\mathbf{T}} \oplus x \oplus y))$? This variety produces the result $\{x, y \in \mathbf{T} \mid x \geq y \geq 0\}$. Since the first equation contains no y variable, y must be less than or equal to x in order to achieve equality between the two (when both are greater than zero).

Now, we can see how the tropical variety is reserved for a specific geometric object, namely the bend locus of a tropical polynomial.

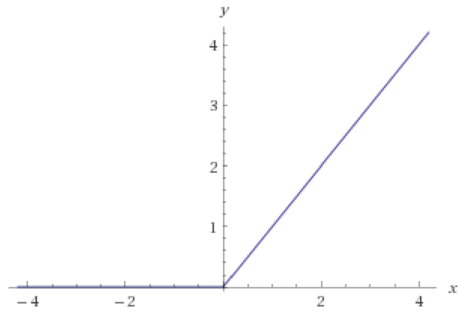
Definition 3.5. To each tropical polynomial $f \in \mathbf{T}[x]$ one can also associate a subset $\mathrm{Bend}(f) \subseteq \mathbf{T}^n$ called the **bend locus** of f. The $\mathrm{Bend}(f)$ is the collection of all points at which the maximum is ahcieved twice. These are the points at which the graph f "bends."

Example 3.6. The following are some bend loci.

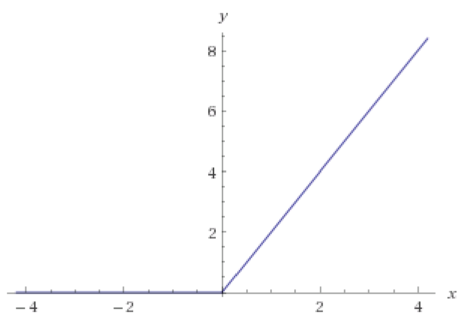
a. $\mathrm{Bend}(x)$: The graph $y = x$ has no bends, so $\mathrm{Bend}(x) = \emptyset$.



b. Bend($1_{\mathbf{T}} \oplus x$): The graph $y = \max\{0, x\}$ has one bend at $x = 0$, so Bend($1_{\mathbf{T}} \oplus x$) $= \{1_{\mathbf{T}}\}$.



c. Bend($1_{\mathbf{T}} \oplus x^{\odot 2}$): The graph $y = \max\{0, 2x\}$ has one bend at $x = 0$, so Bend($1_{\mathbf{T}} \oplus x^{\odot 2}$) $= \{1_{\mathbf{T}}\}$.



3.3. **The Tropical Algebra-Geometric Correspondence.**

Given $S \subseteq \mathbf{T}^n$, let

$$\mathbf{E}(S) = \{(f, g) \mid f, g \in \mathbf{T}[x_1, ..., x_n], f(a) = g(a), a \in S\}.$$

This collection of tropical polynomials and subsets has many nice properties. First, it has the properties of an equivalence relation: reflexive, symmetric and transitive. The other important properties extend to addition and multiplication: if $(f, g), (f', g') \in \mathbf{E}$, then $(f + f'), (g + g'), (ff'), (gg') \in \mathbf{E}$. It turns out that such a subset of a semiring has a name.

Definition 3.7. A **congruence** on a semiring R is a subset $E \subseteq R \times R$ with the following properties:

i. For every $a \in R$, we have $(a, a) \in E$;
ii. If $(a, b) \in E$, then $(b, a) \in E$;
iii. If $(a, b), (b, c) \in E$, then $(a, c) \in E$;
iv. If $(a, b), (a', b') \in E$, then $(a + a', b + b') \in E$;
v. If $(a, b), (a', b')] \in E$, then $(aa', bb') \in E$.

Notice that a congruence is a subsemiring of $R \times R$ that also defines an equivalence relation on R. Congruences on the semiring $\mathbf{T}[x_1, \ldots, x_n]$ are the natural replacement for ideals on the ring $k[x_1, \ldots, x_n]$.

In this chapter, we will explore the correspondence between ideals and varieties. The questions posed at the end of Chapter 1 will begin to take shape as we delve into the Tropical Nullstellensatz for Congruences, a way to identify exactly which congruences correspond to varieties. An important note to make is that the transformation from ideals to varieties is not one-to-one, a reason why Nullstellensatz is important. Different ideals can produce the same variety. For example, $\langle x \rangle$ and $\langle x^2 \rangle$ are different ideals in $\mathbf{C}[x]$ which have the same variety $\mathbf{V}(x) = \mathbf{V}(x^2) = \{0\}$. More serious problems can occur when the field k is not algebraically closed. In these scenarios, different polynomials can produce different ideals but each polynomial can have no real roots, so that the corresponding varieties are empty. Fortunately and wonderfully, in a polynomial ring over an algebraically closed field, the only ideal which represents the empty variety is the entire polynomial ring itself! This is the Weak Nullstellensatz.

4.1. **The Weak Nullstellensatz.**

Theorem 4.1 (The Weak Nullstellensatz)**.** *Let k be an algebraically closed field and $I \subset k[x_1, ..., x_n]$ be an ideal satisfying $\boldsymbol{V}(I) = \emptyset$. Then $I = [x_1, ..., x_n]$.*

It turns out that the tropical analogue of the above result is the following:

Proposition 4.2. [BE15, Thm. 2, pg. 2] *Suppose $f, g \in \boldsymbol{T}[x]$ and $\boldsymbol{V}(f, g) = \emptyset$. Then there exists $h \in \boldsymbol{T}[x]$ with nonzero constant term such that $(h, \epsilon \odot h) \in \langle (f, g) \rangle$ for some real number $\epsilon > 0$.*

The proof of this proposition involves finding an algebraic characterization for when two tropical polynomials define the same tropical function. Given a tropical polynomial f, let $[f]$ denote the function $\mathbf{T}^n \to \mathbf{T}$ defined by f. We use the bracket notation to suggest the idea of an equivalence class of tropical polynomials.

Definition 4.3. The "minimal" representative of $[f]$ is the unique tropical polynomial f_{dsat} whose coefficients are each minimal among all tropical polynomials representing the function $[f]$. We call f_{dsat} the **desaturated** representative of $[f]$.

Many tropical polynomials can produce the same bend loci, and the smallest polynomial to render the bend locus is the desaturated representative. The minimal coefficients needed to produce the graph of $[f]$ is the desaturated polynomial. To better understand this, let's do an example.

Example 4.4. Fix some $t > 1_{\mathbf{T}}$, and consider the desaturated tropical polynomial $f(x) = x \oplus t$. Its square is the polynomial $f^{\odot 2}(x) = x^{\odot 2} \oplus (t \odot x) \oplus t^{\odot 2}$, whose graph is given below. Thus, $(f^{\odot 2})_{\text{dsat}}(x) = x^{\odot 2} \oplus t^{\odot 2}$.

Now let's look at saturated polynomials.

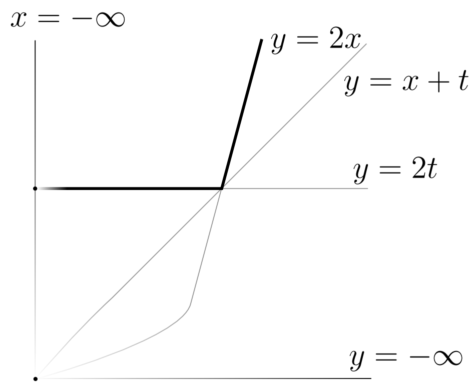
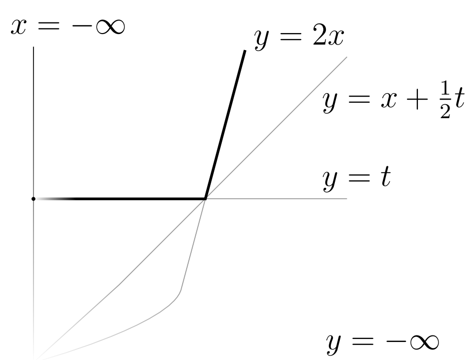


FIGURE 2. The graph of $f^{\odot 2}(x) = x^{\odot 2} \oplus (t \odot x) \oplus t^{\odot 2}$ (in bold).

Definition 4.5. The "maximal" representative of $[f]$ is the unique tropical polynomial f_{sat} whose coefficients are maximal among all tropical polynomials representing $[f]$. We call f_{sat} the saturated representative of $[f]$.

As seen by the definition, saturated polynomials are not only harder to compute, but they are more complicated in general. Understanding the meaning of saturation is the first step. A topical polynomial is saturated if there are all tropical variables of every power less than the maximum. So, the polynomial is completely full with coefficients, unlike desaturated polynomials which contain the least amount of coefficients. These polynomials can be analagous to a simple item like a sponge. When the sponge is saturated with water, it bulges and cannot hold any more. When the sponge is desaturated, it contains only the bare minimum to retain its shape. Let's see an example.

Example 4.6. Fix some $t > 1_{\mathbf{T}}$, and consider the desaturated tropical polynomial $f(x) = x^{\odot 2} \oplus t$. One can show that $f_{\mathrm{sat}}(x) = x^{\odot 2} \oplus (t^{\odot 1/2} \odot x) \oplus t$. This is shown in the graph below:



With this idea of saturation and desaturation comes questions about congruences. If $\mathbf{V}(f, g) = \emptyset$, we want to describe $\langle (f, g) \rangle$ in algebraic terms. We can do this with congruences and further applications of the Nullstellensatz.

4.2. **The Hilbert Nullstellensatz.**

In the classical setting, the issue of understanding $\mathbf{I}(\mathbf{V}(I))$ is answered by Hilbert's famous "Nullstellensatz:"

Theorem 4.7 (The Hilbert Nullstellensatz)**.** [H77, Prop. 1.3A, pg. 4] *Let k be an algebraically closed field. If $f_1, ..., f_s \in k[x_1, ..., x_n]$ are such that $f \in \boldsymbol{I}(\boldsymbol{V}(f_1, ..., f_s))$ then there exists an integer $m \geq 1$ such that $f^m \in \langle f_1, ..., f_s \rangle$ and conversely.*

Given an ideal I in a ring R, the set $\{f \in R \mid f^n \in I \text{ for some } n \geq 1\}$ is called the **radical** of I, and denoted $\sqrt{I}$. Using this terminology, Hilbert's Nullstellensatz says that $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$. We now consider the tropical analogue.

In the tropical situation for a congruence E, it is easy to see that if $(f^{\odot n}, g^{\odot n}) \in E$ for some $n \geq 1$, then $(f, g) \in \mathbf{E}(\mathbf{V}(E))$. One can also show that if $(f, g)^{\times n} \in E$, then $(f, g) \in \mathbf{E}(\mathbf{V}(E))$. Here we are using the so-called "twisted product," defined generally on a semiring R by

$$(a, b) \times (c, d) = (ac + bd, ad + bc).$$

This is the product that results if one imagines representing a pair (a, b) by the difference $a - b$, working in analogy with a ring. One can show that this twisted product has the following nice property:

$$\mathbf{V}((a, b) \times (c, d)) = \mathbf{V}(a, b) \cup \mathbf{V}(c, d).$$

It is a fact that if either $(f^{\odot n}, g^{\odot n}) \in E$ or $(f, g)^{\times n} \in E$ for some $n \geq 1$, then $((f \oplus g)^{\otimes N}, 0_{\mathbf{T}}) \times (f, g) \in E$ for sufficiently large N. So if $((f \oplus g)^{\odot N}, 0_{\mathbf{T}}) \times (f, g) \in E$, then $(f, g) \in \mathbf{E}(\mathbf{V}(E))$. This leads us to a definition.

Definition 4.8. Consider the collection of all pairs (f, g) for which the following condition holds: there exist $r \in \mathbf{T}[x], \epsilon \neq 1_{\mathbf{T}}$ and $N > 0$ such that $((f \oplus g)^{\odot N} \oplus r, \epsilon \odot ((f \oplus g)^{\odot N} \oplus r)) \times (f, g) \in E$. We call this collection the **pre-radical** of E and denote it $\text{rad}^{-}(E)$. Note that $E \subseteq \text{rad}^{-}(E) \subseteq \mathbf{E}(\mathbf{V}(E))$; however, it is unlikely that $\text{rad}^{-}(E)$ is a congruence.

A couple of nice properties comes with this new set. One of which is the fact that $\mathbf{V}(\text{rad}^{-}(E)) = \mathbf{V}(E)$ for any congruence E on $\mathbf{T}[x]$. One can restate our earlier weak tropical Nullstellensatz using this new terminology:

Theorem 4.9. [BE15, Thm. 3, pg. 3] *If E is a finitely-generated congruence on $\boldsymbol{T}[x]$, then $\boldsymbol{V}(E)$ is empty if and only if $(1_{\boldsymbol{T}}, 0_{\boldsymbol{T}}) \in \text{rad}^{-}(E)$.*

Similar to the fact that $\mathbf{V}(I) = \emptyset$ if and only if $1 \in I$, we can now extend this to the statement that $\mathbf{V}(E) = \emptyset$ if and only if $(1_{\mathbf{T}}, 0_{\mathbf{T}}) \in \text{rad}^{-}(E)$. It is also possible to define a **radical** of a congruence, although the definition is fairly complicated. We refer the interested reader to [BE15] for the details.

5. Areas for Future Research

A congruence can be tricky to understand, mainly because of the transitive property. In particular, it makes it difficult to understand the form of a general element of a congruence, even in a congruence generated by a finite set of pairs $(f_1, g_1), \ldots, (f_r, g_r)$. Here is the problem posed:

Question. Suppose E is a finitely generated congruence on $\mathbf{T}[x]$. Suppose there is a "chain" of pairs $(f_1, g_1), (f_2, g_2), \ldots \in E$ with the following two properties:

1) $\langle (f_1, g_1) \rangle \subseteq \langle (f_2, g_2) \rangle \subseteq \ldots \subseteq E$; and
2) $\lim_{i\to\infty} f_i = f$ and $\lim_{i\to\infty} g_i = g$.

Is it true that $(f, g) \in E$?

The answer is not obvious, even in the seemingly simple case when $E = \langle (h, k) \rangle$ and the ascending chain is $(x \oplus t_1, t_1), (x \oplus t_2, t_2), ... \in E$ with $t_1 > t_2 > ... > 1_{\mathbf{T}}$. (One can easily check $\langle (x \oplus t_1, t_1) \rangle \subseteq \langle (x \oplus t_2, t_2) \rangle \subseteq \ldots$) An answer to the question of whether or not $(x, 1_{\mathbf{T}}) \in E$ in this case would give good insight into what is going on in the larger question above.

References

[BE15] A. Bertram and R. Easton, *The tropical Nullstellensatz for congruences*, preprint (2015), 34pp.
[BS14] E. Bruagllé and K. Shaw, *A bit of tropical geometry*, preprint (2014), 27pp.
[G06] A. Gathmann, *Tropical algebraic geometry*, Jahresber. Deutsch. Math.-Verein. **108** (2006), no. 1, 3–32.
[H77] R. Hartshorne, *Algebraic Geometry*, Spring-Verlag (1977), 496pp.
[MS13] D. Maclagan and B. Sturmfels, *Introduction to Tropical Geometry*, in preparation (2013).