Math 344: Homework 6 Solutions

Exercise 1. Practice computing Fourier coefficients.
Let $f(t) = \sin(2\pi t) - \cos(2\pi t) + 5\sin(6\pi t)$.

a) Using the identities $\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ and $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$, determine $\hat{f}(n)$ without doing any integrals.

b) Using the integral formula, compute $\hat{f}(n)$ for all n. (This will be painful.)

Solution.

a) Using our identities, we have

$$\begin{aligned} f(t) &= \sin(2\pi t) - \cos(2\pi t) + 5\sin(6\pi t) \\ &= \frac{e^{2\pi it} - e^{-2\pi it}}{2i} - \frac{e^{2\pi it} + e^{-2\pi it}}{2} + 5 \cdot \frac{e^{6\pi it} - e^{-6\pi it}}{2i} \\ &= -\frac{5}{2i}e^{-6\pi it} + \left(-\frac{1}{2} - \frac{1}{2i}\right)e^{-2\pi it} + \left(-\frac{1}{2} + \frac{1}{2i}\right)e^{2\pi it} + \frac{5}{2i}e^{6\pi it} \\ &= \frac{5i}{2}e^{-6\pi it} + \left(-\frac{1}{2} + \frac{i}{2}\right)e^{-2\pi it} + \left(-\frac{1}{2} - \frac{i}{2}\right)e^{2\pi it} - \frac{5i}{2}e^{6\pi it} \end{aligned}$$

It immediately follows that

$$\hat{f}(-3) = \frac{5}{2}i, \quad \hat{f}(-1) = -\frac{1}{2} + \frac{1}{2}i, \quad \hat{f}(1) = -\frac{1}{2} - \frac{1}{2}i, \quad \hat{f}(3) = -\frac{5}{2}i,$$

and all other $\hat{f}(n) = 0$.

b) By our integral formula, we have

$$\begin{aligned} \hat{f}(0) &= \int_0^1 f(t)\,\mathrm{d}t = \int_0^1 \sin(2\pi t) - \cos(2\pi t) + 5\sin(6\pi t)\,\mathrm{d}t \\ &= \left[-\frac{1}{2\pi}\cos(2\pi t) - \frac{1}{2\pi}\sin(2\pi t) - \frac{5}{6\pi}\cos(6\pi t)\right]_0^1 \\ &= 0. \end{aligned}$$

For all $n \neq 0$, we have

$$\begin{aligned} \hat{f}(n) &= \int_0^1 f(t)e^{-2\pi int}\,\mathrm{d}t = \int_0^1 (\sin(2\pi t) - \cos(2\pi t) + 5\sin(6\pi t))e^{-2\pi int}\,\mathrm{d}t \\ &= \int_0^1 \sin(2\pi t)e^{-2\pi int}\,\mathrm{d}t - \int_0^1 \cos(2\pi t)e^{-2\pi int}\,\mathrm{d}t + 5\int_0^1 \sin(6\pi t)e^{-2\pi int}\,\mathrm{d}t. \end{aligned}$$

Each of these three integrals requires two integration by parts to compute. We could compute theme each manually, or we could look up the following general formulas in a table of integrals:

$$\int \sin(at)e^{bt}\ \mathrm{d}t = \frac{1}{a^2+b^2}(b\sin(at) - a\cos(at))e^{bt} + C$$
$$\int \cos(at)e^{bt}\ \mathrm{d}t = \frac{1}{a^2+b^2}(a\sin(at) + b\cos(at))e^{bt} + C.$$

However, there is a small catch: these formulas are only valid when $a^2+b^2 \neq 0$. For example, we have

$$\int_0^1 \sin(2\pi t)e^{-2\pi int}\ \mathrm{d}t = \left[\frac{1}{4\pi^2 - 4\pi^2 n^2}(-2\pi in\sin(2\pi t) - 2\pi\cos(2\pi t))e^{-2\pi int}\right]_0^1 = 0,$$

so long as $4\pi^2 - 4\pi^2 n^2 \neq 0$, i.e., $n \neq \pm 1$. Similarly, for $n \neq \pm 1$ we also have

$$\int_0^1 \cos(2\pi t)e^{-2\pi int}\ \mathrm{d}t = \left[\frac{1}{4\pi^2 - 4\pi^2 n^2}(2\pi\sin(2\pi t) - 2\pi in\cos(2\pi t))e^{-2\pi itn}\right]_0^1 = 0,$$

and for $n \neq \pm 3$ we have

$$\int_0^1 \sin(6\pi t)e^{-2\pi int}\ \mathrm{d}t = \left[\frac{1}{36\pi^2 - 4\pi^2 n^2}(-2\pi in\sin(6\pi t) - 6\pi\cos(6\pi t))e^{-2\pi int}\right]_0^1 = 0.$$

We've thus shown that $\hat{f}(n) = 0$ for all $n \neq \pm 1, \pm 3$. It remains to computer $\hat{f}(n)$ for $n = \pm 1, \pm 3$. For $n = 1$, observe that

$$\begin{aligned}\int_0^1 \sin(2\pi t)e^{-2\pi it}\ \mathrm{d}t &= \int_0^1 \sin(2\pi t)(\cos(2\pi t) - i\sin(2\pi t))\ \mathrm{d}t \\ &= \int_0^1 \sin(2\pi t)\cos(2\pi t)\ \mathrm{d}t - i\int_0^1 \sin^2(2\pi t)\ \mathrm{d}t \\ &= 0 - i\cdot\frac{1}{2} = -\frac{1}{2}i,\end{aligned}$$

where we computed the last integrals by the usual methods of calculus. By a similar method, one computes

$$\int_0^1 \cos(2\pi t)e^{-2\pi it}\ \mathrm{d}t = \int_0^1 \cos^2(2\pi t)\ \mathrm{d}t - i\int_0^1 \sin(2\pi t)\cos(2\pi t)\ \mathrm{d}t = \frac{1}{2},$$

and

$$\int_0^1 \sin(6\pi t)e^{-2\pi it}\ \mathrm{d}t = \int_0^1 \sin(6\pi t)\cos(2\pi t)\ \mathrm{d}t - i\int_0^1 \sin(6\pi t)\sin(2\pi t)\ \mathrm{d}t = 0.$$

Putting these altogether gives

$$\begin{aligned}\hat{f}(1) &= \int_0^1 \sin(2\pi t)e^{-2\pi it}\ \mathrm{d}t - \int_0^1 \cos(2\pi t)e^{-2\pi it}\ \mathrm{d}t + 5\int_0^1 \sin(6\pi t)e^{-2\pi it}\ \mathrm{d}t \\ &= -\frac{1}{2}i - \frac{1}{2} + 5\cdot 0 = -\frac{1}{2} - \frac{1}{2}i.\end{aligned}$$

We could repeat the same painful procedure to compute $\hat{f}(-1)$, or we could recall that $\hat{f}(-1)$ is the complex conjugate of $\hat{f}(1)$, and so $\hat{f}(-1) = -\frac{1}{2} + \frac{1}{2}i$.

We are done as soon as we repeat the above procedure for $n = 3$. We will omit most of the details and just give the necessary integrals and their values. We ultimately find that

$$\begin{aligned}\hat{f}(3) &= \int_0^1 \sin(2\pi t)e^{-6\pi it}\,\mathrm{d}t - \int_0^1 \cos(2\pi t)e^{-6\pi it}\,\mathrm{d}t + 5\int_0^1 \sin(6\pi t)e^{-6\pi it}\,\mathrm{d}t \\ &= 0 - 0 + 5\cdot\frac{-1}{2}i = -\frac{5}{2}i.\end{aligned}$$

As before, we then also have that $\hat{f}(-3) = \frac{5}{2}i$. To summarize, we have found

$$\hat{f}(-3) = \frac{5}{2}i, \quad \hat{f}(-1) = -\frac{1}{2} + \frac{1}{2}i, \quad \hat{f}(1) = -\frac{1}{2} - \frac{1}{2}i, \quad \hat{f}(3) = -\frac{5}{2}i,$$

and all other $\hat{f}(n) = 0$.

R Look at how much easier it was to compute $\hat{f}(n)$ using the identities! Just as with computing Taylor series, sometimes it can really pay off not to rely exclusively on the integral formula to compute Fourier coefficients.

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Exercise 2. More practice computing Fourier coefficients.
Let $f(t)$ denote the "square wave" function we saw in class, which is periodic with period 1 and is defined on $[0,1)$ by

$$f(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \end{cases}$$

a) Verify what we saw in class, namely that $\hat{f}(0) = 0$ and that for $n \neq 0$ we have

$$\hat{f}(n) = \begin{cases} 0, & \text{if } n \text{ even} \\ \frac{2}{\pi i n}, & \text{if } n \text{ odd} \end{cases}$$

b) Also verify the simplified expression for the Fourier series we found in class, namely

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi int} = \frac{4}{\pi}\sum_{\text{odd } n>0} \frac{1}{n}\sin(2\pi nt).$$

Solution.

a) We first compute

$$\hat{f}(0) = \int_0^1 f(t)\,\mathrm{d}t = \int_0^{1/2} 1\,\mathrm{d}t + \int_{1/2}^1 (-1)\,\mathrm{d}t = 0.$$

For $n \neq 0$, we have

$$\hat{f}(n) = \int_0^1 f(t)e^{-2\pi int}\,\mathrm{d}t = \int_0^{1/2} e^{-2\pi int}\,\mathrm{d}t - \int_{1/2}^1 e^{-2\pi int}\,\mathrm{d}t.$$

To compute these, we make the substitution $u = -2\pi int$ (so $\mathrm{d}u = -2\pi in\,\mathrm{d}t$), obtaining

$$\begin{aligned}
\hat{f}(n) &= \left[\frac{1}{-2\pi in}e^{-2\pi int}\right]_{t=0}^{t=1/2} - \left[\frac{1}{-2\pi int}e^{-2\pi int}\right]_{t=1/2}^{t=1} \\
&= -\frac{1}{2\pi in}\left(e^{-\pi in} - 1\right) + \frac{1}{2\pi in}\left(e^{-2\pi in} - e^{-\pi in}\right).
\end{aligned}$$

To simplify this, recall that $e^{-2\pi in} = 1$ for all n, and that $e^{-\pi in} = \left(e^{-\pi i}\right)^n = (-1)^n$. So, the above expression simplifies to

$$\hat{f}(n) = -\frac{1}{\pi in}\left((-1)^n - 1\right).$$

Considering the cases when n is odd and n is even, we thus find that

$$\hat{f}(n) = \begin{cases} 0, & \text{if } n \text{ even} \\ \frac{2}{\pi in}, & \text{if } n \text{ odd} \end{cases}$$

b) We've shown that the Fourier series for $f(t)$ is

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi int} &= \frac{2}{\pi i}\sum_{\text{odd } n}\frac{1}{n}e^{2\pi int} \\
&= \frac{2}{\pi i}\sum_{\text{odd } n>0}\left(\frac{1}{n}e^{2\pi int} + \frac{1}{-n}e^{2\pi i(-n)t}\right) \\
&= \frac{2}{\pi i}\sum_{\text{odd } n>0}\frac{1}{n}\left(e^{2\pi int} - e^{-2\pi int}\right) \\
&= \frac{4}{\pi}\sum_{\text{odd } n>0}\frac{1}{n}\sin(2\pi nt).
\end{aligned}$$

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Exercise 3. Even more practice computing Fourier coefficients.
Let $f(t)$ denote the "triangle wave" function, which is periodic with period 1 and is defined on $[0,1)$ by

$$f(t) = \begin{cases} t, & 0 \leq t < \frac{1}{2} \\ 1 - t, & \frac{1}{2} \leq t < 1 \end{cases}$$

a) Verify that $\hat{f}(0) = \frac{1}{4}$ and that for $n \neq 0$ we have

$$\hat{f}(n) = \begin{cases} 0, & \text{if } n \text{ even} \\ -\frac{1}{\pi^2 n^2}, & \text{if } n \text{ odd} \end{cases}$$

b) Show that

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi int} = \frac{1}{4} - \frac{2}{\pi^2} \sum_{\text{odd } n>0} \frac{1}{n^2} \cos(2\pi nt).$$

c) **[Optional]** Assuming f equals its Fourier series at $t = 1$, show that $\displaystyle\sum_{\text{odd } n>0} \frac{1}{n^2} = \frac{\pi^2}{8}$.

d) **[Optional]** Using (c), show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution.

a) We first compute

$$\hat{f}(0) = \int_0^1 f(t)\,\mathrm{d}t = \int_0^{1/2} t\,\mathrm{d}t + \int_{1/2}^1 (1-t)\,\mathrm{d}t = \frac{1}{4}.$$

For $n \neq 0$, we have

$$\hat{f}(n) = \int_0^1 f(t)e^{-2\pi int}\,\mathrm{d}t = \int_0^{1/2} te^{-2\pi int}\,\mathrm{d}t + \int_{1/2}^1 (1-t)e^{-2\pi int}\,\mathrm{d}t.$$

To compute these, we make the substitution $u = -2\pi int$ (so $\mathrm{d}u = -2\pi in\,\mathrm{d}t$) and use integration by parts, obtaining

$$\begin{aligned}
\hat{f}(n) &= -\frac{1}{4\pi n^2}\int_0^{-\pi in} ue^u\,\mathrm{d}u - \frac{1}{2\pi in}\int_{-\pi in}^{-2\pi in} e^u\,\mathrm{d}u + \frac{1}{4\pi^2 n^2}\int_{-\pi in}^{-2\pi in} ue^u\,\mathrm{d}u \\
&= -\frac{1}{4\pi^2 n^2}\left[ue^u - e^u\right]_0^{-\pi in} + \frac{i}{2\pi n}\left[e^u\right]_{-\pi in}^{-2\pi in} + \frac{1}{4\pi^2 n^2}\left[ue^u - e^u\right]_{-\pi in}^{-2\pi in} \\
&= -\frac{1}{4\pi^2 n^2}\left(-\pi ine^{-\pi in} - e^{-\pi in} + 1\right) + \frac{i}{2\pi n}\left(e^{-2\pi in} - e^{-\pi in}\right) \\
&\quad + \frac{1}{4\pi^2 n^2}\left(-2\pi ine^{-2\pi in} - e^{-2\pi in} + \pi ine^{-\pi in} + e^{-\pi in}\right)
\end{aligned}$$

To simplify this mess, recall that $e^{-2\pi in} = 1$ for all n, and that $e^{-\pi in} = \left(e^{-\pi i}\right)^n = (-1)^n$. So when n is even, the above expression simplifies to

$$\begin{aligned}
\hat{f}(n) &= -\frac{1}{4\pi^2 n^2}(-\pi in - 1 + 1) + \frac{i}{2\pi n}(1-1) + \frac{1}{4\pi^2 n^2}(-2\pi in - 1 + \pi in + 1) \\
&= 0.
\end{aligned}$$

When n is odd, the expression instead simplifies to

$$\begin{aligned}
\hat{f}(n) &= -\frac{1}{4\pi^2 n^2}(\pi in + 1 + 1) + \frac{i}{2\pi n}(1+1) + \frac{1}{4\pi^2 n^2}(-2\pi in - 1 - \pi in - 1) \\
&= -\frac{1}{\pi^2 n^2}.
\end{aligned}$$

b) We've shown that the Fourier series for $f(t)$ is

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi int} &= \frac{1}{4} - \frac{1}{\pi^2}\sum_{\text{odd } n}\frac{1}{n^2}e^{2\pi int}\\ &= \frac{1}{4} - \frac{1}{\pi^2}\sum_{\text{odd } n>0}\left(\frac{1}{n^2}e^{2\pi int} + \frac{1}{(-n)^2}e^{2\pi i(-n)t}\right)\\ &= \frac{1}{4} - \frac{1}{\pi^2}\sum_{\text{odd } n>0}\frac{1}{n^2}\left(e^{2\pi int} + e^{-2\pi int}\right)\\ &= \frac{1}{4} - \frac{2}{\pi^2}\sum_{\text{odd } n>0}\frac{1}{n^2}\cos(2\pi nt).\end{aligned}$$

c) By the original definition, $f(1) = f(0) = 0$. On the other hand, if f agrees with its Fourier series at $t = 1$, then we have

$$0 = f(1) = \frac{1}{4} - \frac{2}{\pi^2}\sum_{\text{odd } n>0}\frac{1}{n^2},$$

from which the result follows.

d) First notice that

$$\sum_{\text{even } n>0}\frac{1}{n^2} = \sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

We therefore have

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{\text{even } n>0}\frac{1}{n^2} + \sum_{\text{odd } n>0}\frac{1}{n^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} + \frac{\pi^2}{8}.$$

It follows that $\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8}$, and the result follows.

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Exercise 4. [Optional] Approximating with Fourier series.
This exercise is for anyone familiar with computer graphing software, such as MATHEMATICA or WOLFRAMALPHA.

a) In Problem 2 you determined the Fourier series for the square wave function. Let $S_N(t)$ denote the "degree N approximation," namely

$$S_N(t) = \frac{4}{\pi}\sum_{\substack{n=-N\\ \text{odd}}}^{N}\frac{1}{n}\sin(2\pi nt).$$

Plot the graph of $S_N(t)$ for $N = 1, 9$, and 99.

b) Read the paragraph in the middle of page 57 of Osgood, on the Gibbs phenomenon.

c) Repeat part (a) for the triangle wave function of Problem 3.

Solution.

a) Using computer software such as DESMOS or MATHEMATICA, we see the following:

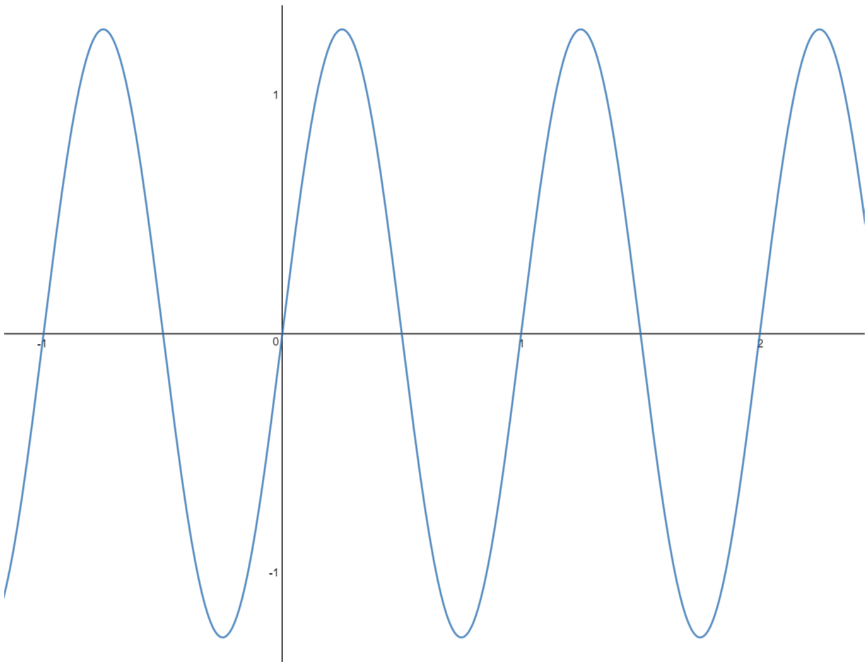

Figure 1: Fourier approximation to square wave, using $N = 1$

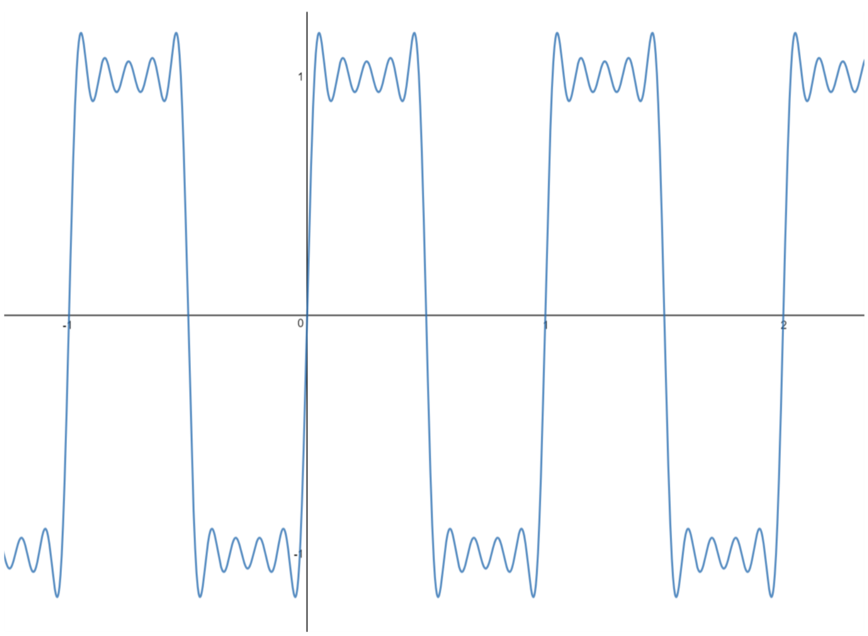

Figure 2: Fourier approximation to square wave, using $N = 9$

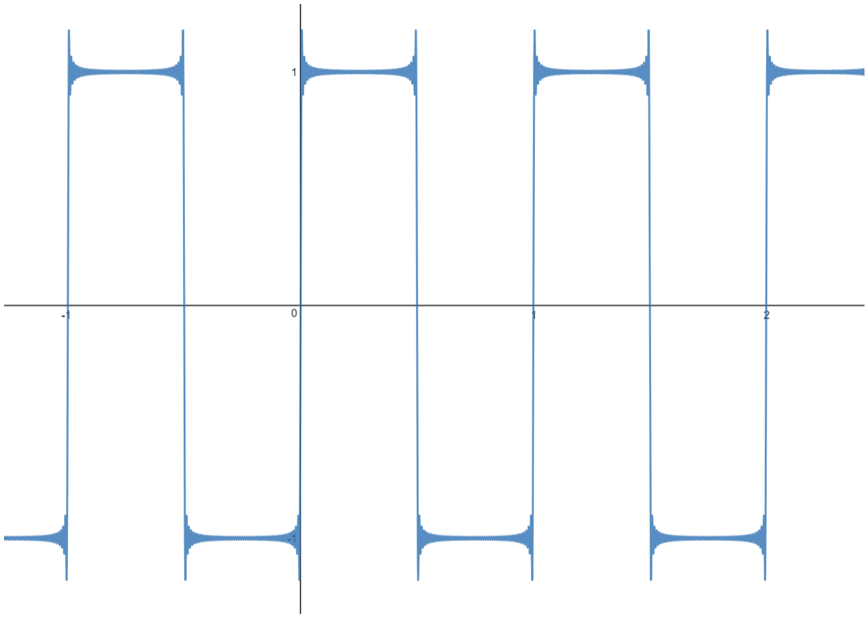

Figure 3: Fourier approximation to square wave, using $N = 99$

c) Once again using computer software, we see the following:

Figure 4: Fourier approximation to triangle wave, using $N = 1$

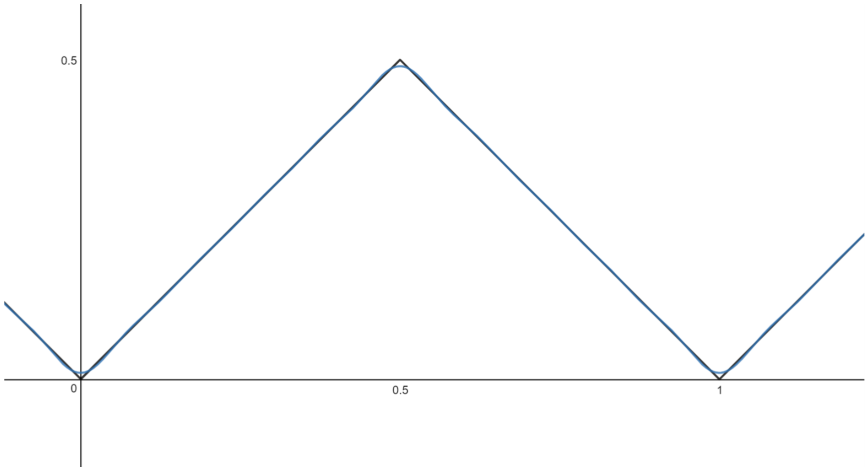

Figure 5: Fourier approximation to triangle wave, using $N = 9$

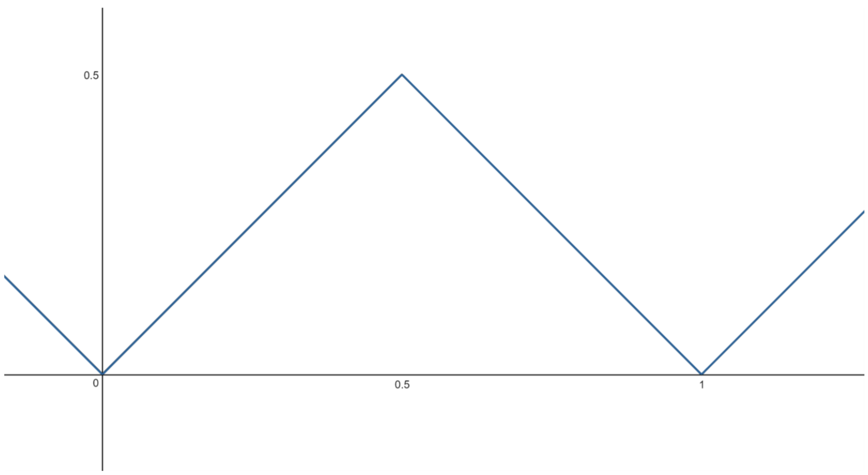

Figure 6: Fourier approximation to triangle wave, using $N = 99$

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Exercise 5. Working with functions with period $T > 0$.
For periodic functions with period $T > 0$, we define an inner product by

$$(f,g) = \int_0^T f(t)\overline{g(t)}\,\mathrm{d}t.$$

Also define a norm function $\|\cdot\|$ by $\|f\|^2 = (f,f)$.

a) Verify that $\left\{e^{2\pi int/T} \mid n = 0, \pm 1, \pm 2, \ldots\right\}$ is an orthogonal set of functions.

b) Show that $\left\|e^{2\pi int/T}\right\| = \sqrt{T}$ for every integer n, so that the set of functions above is in general not an ortho<u>normal</u> set.

c) Define $e_n(t) = \frac{1}{\sqrt{T}} e^{2\pi int/T}$ and show $\{e_n(t) \mid n = 0, \pm 1, \pm 2, \dots\}$ is an orthonormal set.

d) Suppose $f(t)$ is periodic with period $T > 0$. Show that for every integer n we have $\hat{f}(n) = (f, e_n)$, and conclude that the Fourier series for $f(t)$ is $\sum\limits_{n=-\infty}^{\infty} (f, e_n) e_n$.

Solution.

a) Observe that when $m \neq n$ we have

$$\begin{aligned}
\left(e^{2\pi imt/T}, e^{2\pi int/T}\right) &= \int_0^T e^{2\pi imt/T} \cdot e^{-2\pi int/T} \, \mathrm{d}t \\
&= \int_0^T e^{2\pi i(m-n)t/T} \, \mathrm{d}t \\
&= \frac{T}{2\pi i(m-n)} \int_0^{2\pi i(m-n)} e^u \, \mathrm{d}u \quad (\text{using } u = 2\pi i(m-n)t/T) \\
&= \frac{T}{2\pi i(m-n)} \left(e^{2\pi i(m-n)} - e^0\right) \\
&= \frac{T}{2\pi i(m-n)} (1-1) \\
&= 0.
\end{aligned}$$

b) Observe that

$$\left\| e^{2\pi int/T} \right\|^2 = \int_0^T e^{2\pi int/T} \cdot e^{-2\pi int/T} \, \mathrm{d}t = \int_0^T 1 \, \mathrm{d}t = T,$$

so $\left\| e^{2\pi int/T} \right\| = \sqrt{T}$.

c) We could repeat the above calculations with $e_n(t)$, or we could notice that

$$\|e_n(t)\| = \left\| \frac{1}{\sqrt{T}} e^{2\pi int/T} \right\| = \frac{1}{\sqrt{T}} \left\| e^{2\pi int/T} \right\| = \frac{1}{\sqrt{T}} \cdot \sqrt{T} = 1,$$

and for $m \neq m$

$$(e_m(t), e_n(t)) = \left(\frac{1}{\sqrt{T}} e^{2\pi imt/T}, \frac{1}{\sqrt{T}} e^{2\pi int/T} \right) = \frac{1}{T} \left(e^{2\pi imt/T}, e^{2\pi int/T} \right) = \frac{1}{T} \cdot 0 = 0.$$

d) Observe first that

$$(f, e_n) = \int_0^T f(t) \overline{e_n(t)} \, \mathrm{d}t = \int_0^T f(t) \cdot \frac{1}{\sqrt{T}} e^{-2\pi int/T} \, \mathrm{d}t = \hat{f}(n).$$

Now recall that the Fourier series for f is $\sum\limits_{n=-\infty}^{\infty} c_n e^{2\pi int/T}$, where

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-2\pi int/T} \, \mathrm{d}t = \frac{1}{\sqrt{T}} (f, e_n).$$

Putting things together, then, we see that the Fourier series for f is

$$\sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{T}} (f, e_n) e^{2\pi int/T} = \sum_{n=-\infty}^{\infty} (f, e_n) e_n.$$

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Exercise 6. Rayleigh's identity.
Suppose $f(t)$ is a periodic function with period $T > 0$ that equals its Fourier series.

a) Using Problem 5, show that

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |(f, e_n)|^2.$$

b) Using (a), show Rayleigh's identity:

$$\int_0^T |f(t)|^2 \, \mathrm{d}t = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

Solution.

a) We use the various properties of the inner product:

$$\begin{aligned}
\|f\|^2 &= (f, f) \\
&= \left(\sum_{n=-\infty}^{\infty} (f, e_n) e_n, f \right) && \text{(substituting in the Fourier series)} \\
&= \sum_{n=-\infty}^{\infty} ((f, e_n) e_n, f) && \text{(using additivity in the first component)} \\
&= \sum_{n=-\infty}^{\infty} (f, e_n)(e_n, f) && \text{(factoring out the scalar from the first component)} \\
&= \sum_{n=-\infty}^{\infty} (f, e_n)\overline{(f, e_n)} && \text{(using the conjugate symmetry property)} \\
&= \sum_{n=-\infty}^{\infty} |(f, e_n)|^2.
\end{aligned}$$

b) This follows immediately from (a) and our work in Exercise 5:

$$\int_0^T |f(t)|^2 \, \mathrm{d}t = \|f\|^2 = \sum_{n=-\infty}^{\infty} |(f, e_n)|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

R This equality is sometimes stated as "the energy of a signal equals the sum of the energies of its harmonics."

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