Math 344: Homework 6

Complete by Tuesday, October 30

1. **Practice computing Fourier coefficients.** Let $f(t) = \sin(2\pi t) - \cos(2\pi t) + 5\sin(6\pi t)$.

   a) Using the identities $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$, determine $\hat{f}(n)$ without doing any integrals.

   b) Using the integral formula, compute $\hat{f}(n)$ for all n. (This will be painful.)

2. **More practice computing Fourier coefficients.** Let $f(t)$ denote the "square wave" function we saw in class, which is periodic with period 1 and is defined on $[0, 1)$ by

$$f(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \end{cases}$$

   a) Verify what we saw in class, namely that $\hat{f}(0) = 0$ and that for $n \neq 0$ we have

$$\hat{f}(n) = \begin{cases} 0, & \text{if } n \text{ even} \\ \frac{2}{\pi i n}, & \text{if } n \text{ odd} \end{cases}$$

   b) Also verify the simplified expression for the Fourier series we found in class, namely

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n t} = \frac{4}{\pi} \sum_{\text{odd } n>0} \frac{1}{n} \sin(2\pi n t).$$

   R It is not a coincidence that $f(t)$ is an odd function and its Fourier series contains only sine terms.

3. **Even more practice computing Fourier coefficients.** Let $f(t)$ denote the "triangle wave" function, which is periodic with period 1 and is defined on $[0, 1)$ by

$$f(t) = \begin{cases} t, & 0 \le t < \frac{1}{2} \\ 1 - t, & \frac{1}{2} \le t < 1 \end{cases}$$

   a) Verify that $\hat{f}(0) = \frac{1}{4}$ and that for $n \neq 0$ we have

$$\hat{f}(n) = \begin{cases} 0, & \text{if } n \text{ even} \\ -\frac{1}{\pi^2 n^2}, & \text{if } n \text{ odd} \end{cases}$$

   b) Show that $\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n t} = \frac{1}{4} - \frac{2}{\pi^2} \sum_{\text{odd } n>0} \frac{1}{n^2} \cos(2\pi n t)$.

   R It is not a coincidence that $f(t)$ is an even function and its Fourier series contains only cosine terms.

c) **[Optional]** Assuming f equals its Fourier series at $t = 1$, show that $\displaystyle\sum_{\text{odd } n>0} \frac{1}{n^2} = \frac{\pi^2}{8}$.

d) **[Optional]** Using (c), show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

4. **[Optional] Approximating with Fourier series.** This exercise is for anyone familiar with computer graphing software, such as MATHEMATICA or WOLFRAMALPHA.

   a) In Problem 2 you determined the Fourier series for the square wave function. Let $S_N(t)$ denote the "degree N approximation," namely

   $$S_N(t) = \frac{4}{\pi} \sum_{\substack{n=-N \\ \text{odd}}}^{N} \frac{1}{n} \sin(2\pi n t).$$

   Plot the graph of $S_N(t)$ for $N = 1, 9$, and 99.

   b) Read the paragraph in the middle of page 57 of Osgood, on the Gibbs phenomenon.

   c) Repeat part (a) for the triangle wave function of Problem 3.

5. **Working with functions with period $T > 0$.** For periodic functions with period $T > 0$, we define an inner product by

   $$(f, g) = \int_0^T f(t)\overline{g(t)}\,\mathrm{d}t.$$

   Also define a norm function $\|\cdot\|$ by $\|f\|^2 = (f, f)$.

   a) Verify that $\left\{e^{2\pi i n t/T} \mid n = 0, \pm 1, \pm 2, \ldots\right\}$ is an orthogonal set of functions.

   b) Show that $\left\|e^{2\pi i n t/T}\right\| = \sqrt{T}$ for every integer n, so that the set of functions above is in general not an ortho<u>normal</u> set.

   c) Define $e_n(t) = \frac{1}{\sqrt{T}} e^{2\pi i n t/T}$ and show $\{e_n(t) \mid n = 0, \pm 1, \pm 2, \ldots\}$ is an orthonormal set.

   d) Suppose $f(t)$ is periodic with period $T > 0$. Show that for every integer n we have $\hat{f}(n) = (f, e_n)$, and conclude that the Fourier series for $f(t)$ is $\displaystyle\sum_{n=-\infty}^{\infty} (f, e_n) e_n$.

6. **Rayleigh's identity.** Suppose $f(t)$ is a periodic function with period $T > 0$ that equals its Fourier series.

   a) Using Problem 5, show that $\|f\|^2 = \displaystyle\sum_{n=-\infty}^{\infty} |(f, e_n)|^2$.

   Hint: See page 33 in Osgood for the argument in the case $T = 1$.

   b) Using (a), show Rayleigh's identity: $\displaystyle\int_0^T |f(t)|^2\,\mathrm{d}t = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$.