Math 344: Quiz 2 Solutions

Fall 2018

Consider the differential equation $x^2y'' - \frac{1}{3}x(x+8)y' + 2y = 0$.

Problem 1. Why we do not expect the general solution to be represented by a power series centered at $x_0 = 0$?

Solution. In standard form this differential equation is $y'' - \frac{x+8}{3x}y' + \frac{2}{x^2}y = 0$. Since the coefficient functions are not analytic at $x_0 = 0$ it follows that $x_0 = 0$ is not an ordinary point of the differential equation. (It is a regular singular point, so we should look for a Frobenius series solution.) ■

Problem 2. Find two linearly independent Frobenius series solutions. Give at least three nonzero terms in each series.

Solution. Substituting $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ (with $a_0 \neq 0$) into the differential equation and simplifying[1] yields

$$\left(r^2 - \frac{11}{3}r + 2\right)a_0 + \sum_{m=1}^{\infty}\left[\left((m+r)(m+r-1) - \frac{8}{3}(m+r) + 2\right)a_m - \frac{1}{3}(m+r-1)a_{m-1}\right]x^m = 0.$$

It follows that $y(x)$ is a solution if and only if $r^2 - \frac{11}{3}r + 2 = 0$ and

$$\left((m+r)(m+r-1) - \frac{8}{3}(m+r) + 2\right)a_m - \frac{1}{3}(m+r-1)a_{m-1} = 0, \quad \text{for } m \geq 1.$$

The first equality implies $r = 3, \frac{2}{3}$. (The first equality is also the indicial equation, which we could have solved at the start of this problem.) We now handle each root in turn.

- For $r = 3$, the recurrence relation simplifies to

  $$m(3m+7)a_m - (m+2)a_{m-1} = 0, \quad m \geq 1$$

  Substituting $m = 1, 2, 3, \ldots$ into the above recurrence yields

  $$a_1 = \frac{3}{10}a_0, \quad a_2 = \frac{2}{13}a_1 = \frac{3}{65}a_0, \quad \ldots$$

  Taking $a_0 = 1$, we therefore obtain the solution

  $$y_1(x) = x^3\left(1 + \frac{3}{10}x + \frac{3}{65}x^2 + \cdots\right) = x^3 + \frac{3}{10}x^4 + \frac{3}{65}x^5 + \cdots.$$

  (It also would have been fine to leave a_0 in the series.)

[1] See the next page if you're interested in these calculations.

► For $r = \frac{2}{3}$, the recurrence relation simplifies to

$$3m(3m-7)a_m - (3m-1)a_{m-1} = 0, \quad m \geq 1.$$

Substituting $n = 1, 2, 3, \ldots$ into the above recurrence yields

$$a_1 = \frac{2}{-12}a_0 = -\frac{1}{6}a_0, \quad a_2 = \frac{5}{-6}a_1 = \frac{5}{36}a_0, \quad \ldots$$

Taking $a_0 = 1$, we therefore obtain the solution

$$y_2(x) = x^{\frac{2}{3}}\left(1 - \frac{1}{6}x + \frac{5}{36}x^2 + \cdots\right) = x^{\frac{2}{3}} - \frac{1}{6}x^{\frac{5}{3}} + \frac{5}{36}x^{\frac{8}{3}} + \cdots.$$

(Once again, it would have been fine to leave a_0 in the solution. It also would have been fine to use b_n instead of a_n for the coefficients in this second case, to help distinguish them from the coefficients we found in the first case.)

■

For reference, here are the steps that occur after substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ into the differential equation:

$$x^2y'' - \frac{1}{3}x(x+8)y' + 2y = x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2} - \frac{1}{3}x(x+8)\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1} + 2\sum_{n=0}^{\infty}a_nx^{n+r}$$

$$= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r} - \sum_{n=0}^{\infty}\frac{1}{3}(n+r)a_nx^{n+r+1} - \sum_{n=0}^{\infty}\frac{8}{3}(n+r)a_nx^{n+r} + \sum_{n=0}^{\infty}2a_nx^{n+r}$$

$$= x^r\left[\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^n - \sum_{n=0}^{\infty}\frac{1}{3}(n+r)a_nx^{n+1} - \sum_{n=0}^{\infty}\frac{8}{3}(n+r)a_nx^n + \sum_{n=0}^{\infty}2a_nx^n\right]$$

$$= x^r\left[\sum_{m=0}^{\infty}(m+r)(m+r-1)a_mx^m - \sum_{m=1}^{\infty}\frac{1}{3}(m-1+r)a_{m-1}x^m - \sum_{m=0}^{\infty}\frac{8}{3}(m+r)a_mx^m + \sum_{m=0}^{\infty}2a_mx^m\right]$$

$$= x^r\left[\underbrace{\left(r^2 - \frac{11}{3}r + 2\right)a_0}_{m=0} + \sum_{m=1}^{\infty}\left[\left((m+r)(m+r-1) - \frac{8}{3}(m+r) + 2\right)a_m - \frac{1}{3}(m+r-1)a_{m-1}\right]x^m\right].$$

Setting this equation equal to zero and canceling the factor of x^r yields the equation we claimed at the beginning of the solution.