1. Vocabulary

The **standard form** of a second-order linear homogeneous differential equation is

For a differential equation in the above form, the point $x_0 = 0$ is:

- an **ordinary point** of the differential equation if
- a **regular singular point** of the differential equation if
- an **irregular singular point** of the differential equation if

The general form of a **power series centered at** $x_0 = 0$ is

$y(x) =$

The general form of a **Frobenius series centered at** $x_0 = 0$ is

$y(x) =$

2. Classification of Solutions

For each differential equation below, determine the following:

- the **standard form** of the equation;
- whether $x = 0$ is an **ordinary**, **regular singular**, or **irregular singular** point; and
- whether the general solution can be written in the form $y(x) = c_1 y_1(x) + c_2 y_2(x)$, where:
  - **both** $y_1(x), y_2(x)$ can be represented by **power series**; or
  - **both** $y_1(x), y_2(x)$ can be represented by **Frobenius series**; or
  - **at least one** of $y_1(x), y_2(x)$ can be represented by a **Frobenius series**; or
  - the form of $y_1(x), y_2(x)$ is **unknown**.

Differential Equation	Standard Form	Point Type	Form of $y_1(x), y_2(x)$
$(1+x)y'' + xy' - y = 0$			
$x^2 y'' + xe^{2x} y' - 2y = 0$			
$(2+3x^2)y'' + 20y' - y = 0$			
$x^2 y'' + x(1+2x)y' - \frac{1}{4}y = 0$			
$x^2 y'' + 2y' - xy = 0$			
$xy'' - (3+x)y' + \frac{4-x}{x}y = 0$			
$2x^2 y'' + xy' - (1+x)xy = 0$			

3. Explicit Computations

Problem 1 Consider the differential equation $y'' - x^2 y' - 3xy = 0$. Since $x_0 = 0$ is an ordinary point of this equation, the general solution can be represented by a power series centered at $x_0 = 0$. After substituting in the function $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and simplifying, we obtain

$$\underbrace{2a_2}_{m=0} + \underbrace{(-3a_0 + 6a_3)x}_{m=1} + \sum_{m=2}^{\infty} \left[(m+2)(m+1)a_{m+2} - (m+2)a_{m-1}\right] x^m = 0.$$

Use this information to determine two linearly independent solutions to the differential equation.

Problem 2 The remaining exercises will all be concerned with the differential equation

$$4x^2 y'' - (3 + 4x)y = 0.$$

Since $x_0 = 0$ is a singular point of this differential equation, this equation is not guaranteed to possess a solution that can be represented by a power series centered at $x_0 = 0$. Suppose we didn't realize this and went in search of a power series solution. After substituting in the function $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and simplifying, we obtain

$$\underbrace{-3a_0}_{m=0} + \sum_{m=1}^{\infty} \left[(4m^2 - 4m - 3)a_m - 4a_{m-1}\right] x^m = 0.$$

What happens when you solve for the unknown coefficients?

Problem 3 Since $x_0 = 0$ is a regular singular point we know there is at least one solution that can be represented by a Frobenius series centered at $x_0 = 0$. After substituting in the function $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$ (with $a_0 \neq 0$) and simplifying, we obtain

$$\underbrace{(4r^2 - 4r - 3)a_0}_{m=0} + \sum_{m=1}^{\infty} [(4(m+r)(m+r-1) - 3)a_m - 4a_{m-1}]\, x^m = 0.$$

Use this information to determine a nontrivial solution to the differential equation.

Problem 4 What goes wrong when you attempt to find a second Frobenius series solution?

Problem 5 **(Challenge)** It turns out that the second solution is of the form

$$y_2(x) = Ay_1(x)\ln(x) + x^{-1/2}\sum_{n=0}^{\infty} b_n x^n,$$

where $y_1(x)$ is the Frobenius series solution you found (and we assume $b_0 \neq 0$). After substituting in this function (and much simplification), you eventually obtain

$$\underbrace{(-b_0 - b_1)}_{m=0} + \sum_{m=2}^{\infty}\left[m(m-2)b_m - b_{m-1} + \frac{2A(m-1)}{m!(m-2)!}\right]x^m = 0.$$

(The $m = 1$ term vanishes.) Use this to find a second, linearly independent solution to the differential equation.