Vocabulary

- We say the **limit of** $f(x)$ **as** x **approaches** a **is** L if:

  we can guarantee the values of $f(x)$ are arbitrarily close to L so long as x is sufficiently close to (but different than) a.

  In terms of notation, we write

  $$\lim_{x\to a} f(x) = L.$$

- A function f is **continuous at** a if

  1) $f(a)$ is defined;

  2) $\lim_{x\to a} f(x)$ exists; and

  3) $\lim_{x\to a} f(x) = f(a)$.

- The **derivative** of a function f at a is

  $$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h},$$

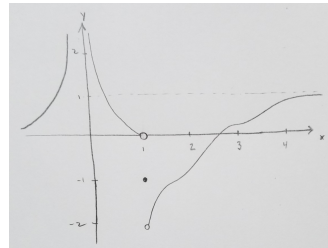
  provided the limit exists.

  In terms of the graph of f, this number is the **slope** of the **tangent** line to the graph at the point $(a, f(a))$.

Concepts

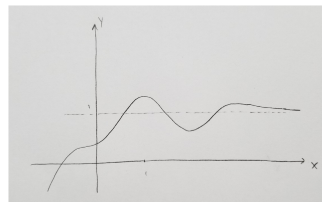
1. Sketch the graph of a function f that has the following properties:

$$\lim_{x\to 0} f(x) = \infty, \quad f(1) = -1,$$
$$\lim_{x\to 1^-} f(x) = 0, \quad \lim_{x\to 1^+} f(x) = -2,$$
$$\lim_{x\to\infty} f(x) = 1$$



SAMPLE SOLUTION

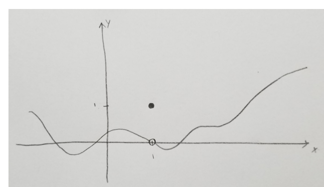
2. Sketch the graph of a continuous function that has a horizontal asymptote at $y = 1$ that the graph crosses exactly three times.



SAMPLE SOLUTION

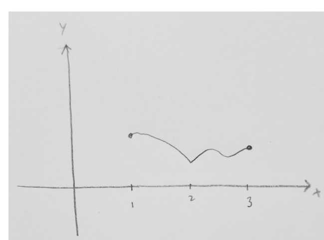
3. If $\lim_{x\to\infty} f(x) = 4$, then the graph has a **horizontal** asymptote at $y = 4$.

4. Sketch the graph of a function that is defined everywhere but is not continuous at $x = 1$.



SAMPLE SOLUTION

5. Sketch the graph of a function f that is continuous on $[1,3]$ but not differentiable at $x = 2$.



SAMPLE SOLUTION

6. **True** or **False**: If $\lim_{x\to 2} f(x) = 3$, then necessarily $f(2) = 3$. FALSE
7. **True** or **False**: If $f(2) = 3$, then necessarily $\lim_{x\to 2} f(x) = 3$. FALSE
8. **True** or **False**: If f is continuous, then necessarily $\lim_{x\to 2} f(x) = f(2)$. TRUE
9. **True** or **False**: If f is continuous at a, then f must be differentiable at a. FALSE
10. **True** or **False**: If f is differentiable at a, then f must be continuous at a. TRUE

Computations

Limits

Exercise 1. Compute the following limits. If a limit does not exist, briefly explain why.

a) $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$ **b)** $\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x}$ **c)** $\lim_{x\to -\infty} \frac{x^2-7x}{x+1}$ **d)** $\lim_{x\to 3^+} \frac{x^2+4}{x-3}$

Solution.

a) We use the conjugate trick to help us compute this limit:

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}} = \lim_{x\to 1} \frac{1-x}{(1-x)(1+\sqrt{x})} = \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}.$$

b) We use some algebra to help us compute this limit:

$$\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \lim_{x\to 0} \frac{\frac{2-(2+x)}{2(2+x)}}{x} = \lim_{x\to 0} \frac{\frac{-x}{2(2+x)}}{x} = \lim_{x\to 0} \frac{-x}{2(2+x)x} = \lim_{x\to 0} \frac{-1}{2(2+x)} = -\frac{1}{4}.$$

c) First notice that when x is a large negative number, we have

$$\frac{x^2-7x}{x+1} \approx \frac{x^2}{x} = x,$$

and so we predict that the given limit equals negative infinity. More formally, we can compute

$$\lim_{x\to -\infty} \frac{x^2-7x}{x+1} = \lim_{x\to -\infty} \frac{x(x-7)}{x(1+\frac{1}{x})} = \lim_{x\to -\infty} \frac{x-7}{1+\frac{1}{x}} = -\infty,$$

where the last equality follows because the numerator tends to negative infinity while the denominator just tends to 1.

d) When x is very close to (but slightly larger than) 3, the numerator is very close to $3^2+4=13$ and the denominator is a very small positive number. Since 13 divided by a very small positive equals a very large positive, we see that

$$\lim_{x\to 3^+} \frac{x^2+4}{x-3} = \infty.$$

■

ADDITIONAL PRACTICE

- ▶ §2.2: 11-42
- ▶ §2.6: 13-36, 69-76

Continuity

Exercise 2. If f is continuous and $3x-3 \leq f(x) \leq x^2-x+1$ for every $x \neq 2$, find $\lim_{x\to 2} f(x)$.

Solution. This is a perfect situation for the Squeeze Theorem. Observe that

$$\lim_{x\to 2}(3x-3) = 3(2)-3 = 3 \quad \text{and} \quad \lim_{x\to 2}(x^2-x+1) = (2)^2-2+1 = 3.$$

By the Squeeze Theorem, it follows that $\lim_{x\to 2} f(x) = 3$. ■

Exercise 3. Where is the function $f(x) = \frac{\sin(x)}{x^2-1}$ continuous?

Solution. We know that both the numerator and denominator are continuous everywhere. It follows that their quotient is continuous wherever the denominator is nonzero. Since $x^2-1 = (x+1)(x-1)$, the denominator is zero exactly when $x = \pm 1$. We can therefore conclude that f is continuous everywhere except $x = \pm 1$ (where it is not defined). ■

Exercise 4. Determine the value of a that makes the function below continuous at 1:

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{for } x \neq 1 \\ a, & \text{for } x = 1. \end{cases}$$

Solution. We know that f is continuous at 1 if and only if $\lim_{x\to 1} f(x) = f(1)$. We are given $f(1) = a$. For the limit, we compute

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-4}{x-2} = \lim_{x\to 1} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 1}(x+2) = 3.$$

So, the function is continuous at 1 exactly when $a = 3$. ■

ADDITIONAL PRACTICE

- §2.5: 29, 30, 37-46

Derivatives

Exercise 5. For each function below, use the limit definition of the derivative to find an equation for the line tangent to the graph at the given point:

a) $f(x) = 1 - 2x^2 \quad (1, -1)$ **b)** $g(x) = -x^3, \quad (-2, 8)$ **c)** $F(t) = \frac{3}{t^2}, \quad (3, \frac{1}{3})$

Solution.

a) The slope of the desired tangent line is $f'(1)$, which we compute first:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1 - 2(1+h)^2) - (-1)}{h} = \lim_{h\to 0} \frac{1 - 2(1 + 2h + h^2) + 1}{h}$$
$$= \lim_{h\to 0} \frac{-4h - 2h^2}{h} = \lim_{h\to 0} \frac{h(-4 - 2h)}{h} = \lim_{h\to 0}(-4 - 2h) = -4.$$

It now follows that the equation of the tangent line is

$$y - (-1) = -4(x - 1).$$

b) Repeating the process in (a), we first compute

$$g'(-2) = \lim_{h\to 0} \frac{g(-2+h) - g(-2)}{h} = \lim_{h\to 0} \frac{-(-2+h)^3 - (8)}{h} = \lim_{h\to 0} \frac{-(-8 + 12h - 6h^2 + h^3) - 8}{h}$$
$$= \lim_{h\to 0} \frac{-12h + 6h^2 - h^3}{h} = \lim_{h\to 0} \frac{h(-12 + 6h - h^2)}{h} = \lim_{h\to 0}(-12 + 6h - h^2) = -12.$$

It now follows that the equation of the tangent line is

$$y - 8 = -12(x - (-2)).$$

c) Once more repeating the process in (a), we first compute

$$F'(3) = \lim_{h\to 0} \frac{F(3+h) - F(3)}{h} = \lim_{h\to 0} \frac{\frac{3}{(3+h)^2} - \frac{1}{3}}{h} = \lim_{h\to 0} \frac{\frac{3\cdot 3 - (3+h)^2}{3(3+h)^2}}{h} = \lim_{h\to 0} \frac{\frac{9 - (9 + 6h + h^2)}{3(3+h)^2}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{-6h - h^2}{3(3+h)^2}}{h} = \lim_{h\to 0} \frac{h(-6 - h)}{3(3+h)^2 h} = \lim_{h\to 0} \frac{-6 - h}{3(3+h)^2} = -\frac{2}{9}.$$

It now follows that the equation of the tangent line is

$$y - \frac{1}{3} = -\frac{2}{9}(x - 3).$$

■

ADDITIONAL PRACTICE

► §3.1: 5-22

Exercise 6. Compute each function's derivative using the limit definition of the derivative:

a) $f(x) = x^2 + 2$ **b)** $g(x) = \sqrt{x+1}$ **c)** $F(x) = \frac{1}{x-1}$

Solution.

a) Using the limit definition of the derivative, we compute

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{((x+h)^2+2)-(x^2+2)}{h} \\ &= \lim_{h\to 0} \frac{(x^2+2xh+h^2+2)-(x^2+2)}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0}(2x+h) = 2x. \end{aligned}$$

b) Repeating the process above, we compute

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}} \\ &= \lim_{h\to 0} \frac{(x+h+1)-(x+1)}{h(\sqrt{x+h+1}+\sqrt{x+1})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+1}+\sqrt{x+1})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}. \end{aligned}$$

c) Repeating the process above once more, we compute

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)-1}-\frac{1}{x-1}}{h} \\ &= \lim_{h\to 0} \frac{\frac{(x-1)-(x+h-1)}{(x-1)(x+h-1)}}{h} = \lim_{h\to 0} \frac{\frac{-h}{(x-1)(x+h-1)}}{h} \\ &= \lim_{h\to 0} \frac{-1}{(x-1)(x+h-1)} = -\frac{1}{(x-1)^2}. \end{aligned}$$

■

ADDITIONAL PRACTICE

- §3.2: 7-26

Differentiation Rules

The Power Rule $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$

Trig Rules $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \cos(x)$

$\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = -\sin(x)$

The Product Rule $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

The Quotient Rule $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

The Chain Rule $(f(u(x)))' = f'(u(x)) \cdot u'(x)$

Exercise 7. Compute the derivative of each function below.

a) $f(x) = x^2\cos(3x-1)$ **b)** $g(t) = \left(\sin(t^2-1)\right)^8$ **c)** $F(\theta) = \frac{\sin(\theta^3)}{\theta^2+1}$

Solution.

a) Recalling the Product Rule and the Chain Rule, we compute

$$f'(x) = 2x \cdot \cos(3x-1) + x^2 \cdot (-\sin(3x-1) \cdot 3) = 2x\cos(3x-1) - 3x^2\sin(3x-1).$$

b) Using the Chain Rule twice, we compute

$$g'(t) = 8(\sin(t^2-1))^7 \cdot \cos(t^2-1) \cdot 2t.$$

c) Using the Quotient Rule and the Chain Rule, we compute

$$F'(\theta) = \frac{(\theta^2+1)(\cos(\theta^3) \cdot 3\theta^2) - (\sin(\theta^3))(2\theta)}{(\theta^2+1)^2} = \frac{3\theta^2(\theta^2+1)\cos(\theta^3) - 2\theta\sin(\theta^3)}{(\theta^2+1)^2}.$$

■

ADDITIONAL PRACTICE

- Chapter 3 Review: 1-7, 9-13, 17-20, 27-40

Implicit Differentiation

Exercise 8. Find the equation of the line tangent to the curve $2x^4 + y^3 = 5x^2 y$ at the point $(1,2)$.

Solution. We first use implicit differentiation to find the slope of the tangent line. We compute

$$8x^3 + 3y^2 \cdot \frac{dy}{dx} = 10xy + 5x^2 \cdot \frac{dy}{dx}.$$

When $x = 1$ and $y = 2$, this gives

$$8(1)^3 + 3(2)^2 \cdot \frac{dy}{dx} = 10(1)(2) + 5(1)^2 \cdot \frac{dy}{dx},$$

or $\frac{dy}{dx} = \frac{12}{7}$. So, the slope of the tangent line at $(1,2)$ is $\frac{12}{7}$, and hence the equation of that line is $y - 2 = \frac{12}{7}(x-1)$. ■

Exercise 9. If $\sqrt{x} + \sqrt{y} = 1$, find $\frac{d^2y}{dx^2}$.

Solution. Taking the derivative (with respect to x) of both sides of the given equation, we find that

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0,$$

which gives $\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$. We can then compute

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{x} \cdot \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} - \sqrt{y}\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

Plugging $\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$ into this equation then gives

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{x} \cdot \frac{1}{2\sqrt{y}} \cdot \frac{-\sqrt{y}}{\sqrt{x}} - \sqrt{y}\frac{1}{2\sqrt{x}}}{x} = -\frac{-\frac{1}{2} - \frac{\sqrt{y}}{2\sqrt{x}}}{x} = \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}} = \frac{1}{2x\sqrt{x}}.$$

(You didn't have to do the last simplification, which used the fact that $\sqrt{x} + \sqrt{y} = 1$.) ■

ADDITIONAL PRACTICE

- §3.7: 1-28
- Chapter 3 Review: 41-48