Math 344: Homework 7 Solutions

Exercise 1. Orthogonality and Length for Functions.

Let $g(x) = e^{2\pi ix}$ and $h(x) = 2i\sin(6\pi x)$. Assume the domain for both functions is $[0,1]$.

a) Determine whether or not the functions g and h are orthogonal, using the inner product we defined in class. (Consider converting h into exponential functions.)

b) Compute the lengths of g and h, using the notion of length we defined in class.

Solution.

a) We compute

$$\begin{aligned}(g,h) &= \int_0^1 e^{2\pi ix}\cdot(-2i\sin(6\pi x)\,\mathrm{d}x = -\int_0^1 e^{2\pi ix}\cdot\left(e^{6\pi ix}-e^{-6\pi ix}\right)\mathrm{d}x \\ &= -\int_0^1 e^{8\pi ix}-e^{-4\pi ix}\,\mathrm{d}x = -\left[\frac{1}{8\pi i}e^{8\pi ix}+\frac{1}{4\pi i}e^{-4\pi ix}\right]_0^1 = 0.\end{aligned}$$

So, the functions are orthogonal.

b) We compute

$$\|g\|^2 = (g,g) = \int_0^1 e^{2\pi ix}\cdot e^{-2\pi ix}\,\mathrm{d}x = \int_0^1 1\,\mathrm{d}x = 1,$$

so $\|g\| = 1$. Similarly, we compute

$$\|h\|^2 = (h,h) = \int_0^1 2i\sin(6\pi x)\cdot(-2i\sin(6\pi x))\,\mathrm{d}x = 4\int_0^1 \sin^2(6\pi x)\,\mathrm{d}x.$$

To finish this computation we can either use a half-angle identity, or convert to exponentials. Using a half-angle identity, we would compute

$$\begin{aligned}4\int_0^1 \sin^2(6\pi x)\,\mathrm{d}x &= 4\int_0^1 \frac{1}{2}(1-\cos(12\pi x))\,\mathrm{d}x = 2\int_0^1 1-\cos(12\pi x)\,\mathrm{d}x \\ &= 2\left[x-\frac{1}{12\pi}\sin(12\pi x)\right]_0^1 = 2,\end{aligned}$$

and so $\|h\| = \sqrt{2}$. If we had instead converted to exponentials, we would have computed

$$\begin{aligned}
4\int_0^1 \sin^2(6\pi x)\,\mathrm{d}x &= 4\int_0^1 \left(\frac{e^{6\pi ix} - e^{-6\pi ix}}{2i}\right)^2 \mathrm{d}x \\
&= -\int_0^1 e^{12\pi ix} - 2 + e^{-12\pi ix}\,\mathrm{d}x \\
&= -\left[\frac{1}{12\pi i}e^{12\pi ix} - 2t + \frac{1}{-12\pi i}e^{-12\pi ix}\right]_0^1 \\
&= -\left[\left(\frac{1}{12\pi i} - 2 - \frac{1}{12\pi i}\right) - \left(\frac{1}{12\pi i} - \frac{1}{12\pi i}\right)\right] \\
&= 2,
\end{aligned}$$

and so again $\|h\| = \sqrt{2}$.

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Exercise 2. Plugging Fourier series into PDEs.

Suppose $u(x,t)$ is a function that is periodic in t with period 1 and satisfies the differential equation $u_{tt}(x,t) + u_{xx}(x,t) = 0$. Write down (but do not solve) a differential equation satisfied by the Fourier coefficients for $u(x,t)$.

Solution. Since $u(x,t)$ is periodic in t with period 1, it has a Fourier series representation

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(x)e^{2\pi int}.$$

Substituting into the differential equation we obtain

$$\frac{\partial^2}{\partial t^2}\left(\sum_{n=-\infty}^{\infty} c_n(x)e^{2\pi int}\right) + \frac{\partial^2}{\partial x^2}\left(\sum_{n=-\infty}^{\infty} c_n(x)e^{2\pi int}\right) = 0.$$

Passing the partial derivatives inside the summations and combining the resulting sums, we see that

$$\sum_{n=-\infty}^{\infty} ((2\pi in)^2 c_n(x) + c_n''(x))e^{2\pi int} = 0.$$

It follows that for every n we have

$$(2\pi in)^2 c_n(x) + c_n''(x) = 0,$$

or equivalently

$$c_n''(x) - 4\pi^2 n^2 c_n(x) = 0.$$

R Observe that each function $c_n(x)$ is a solution to a second-order, linear, homogeneous differential equation with constant coefficients. Using our Math 244 skills, we could deduce that $c_n(x)$ must be of the form $c_n(x) = c_{n,1}e^{2\pi|n|x} + c_{n,2}e^{-2\pi|n|x}$ for some constants $c_{n,1}, c_{n,2}$.

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Exercise 3. Solving PDEs using Fourier series.

Suppose $u(x,t)$ is a function that is periodic in x with period 1 and satisfies the differential equation $u_t(x,t) = u_x(x,t)$. Determine the Fourier series representation of $u(x,t)$.

Solution. Since $u(x,t)$ is periodic in x with period 1, it has a Fourier series representation

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi inx}.$$

Substituting into the given differential equation yields

$$\frac{\partial}{\partial t}\left(\sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi inx}\right) = \frac{\partial}{\partial x}\left(\sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi inx}\right).$$

Passing the derivatives inside the summations, we obtain

$$\sum_{n=-\infty}^{\infty} c_n'(t) e^{2\pi inx} = \sum_{n=-\infty}^{\infty} c_n(t) \cdot (2\pi in) e^{2\pi inx}.$$

Matching coefficients, we see that $c_n'(t) = 2\pi in \cdot c_n(t)$ for every n. Solving this ordinary differential equation yields $c_n(t) = c_n(0)e^{2\pi int}$. Thus, the solution $u(x,t)$ is

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(0) e^{2\pi int} e^{2\pi inx}.$$

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