Math 344: Homework 8 Solutions

The Fourier Transform

Exercise 1. Directly computing a Fourier transform.

Consider the function defined below:

$$f(t) = \begin{cases} 0, & \text{if } |t| > 1 \\ -t, & \text{if } -1 \le t < 0 \\ t, & \text{if } 0 \le t \le 1 \end{cases}$$

This is an even real-valued function, so its Fourier transform is guaranteed to also be an even real-valued function. Compute the Fourier transform of f and write your final answer in the form of a real-valued function, i.e., using only real numbers.

Solution. Using our integral definition of the Fourier transform, we compute

$$\begin{aligned}
(\mathscr{F} f)(s) &= \int_{-\infty}^{\infty} f(t) e^{-2\pi its}\,\mathrm{d}t \\
&= \int_{-1}^{0} -te^{-2\pi its}\,\mathrm{d}t + \int_{0}^{1} te^{-2\pi its}\,\mathrm{d}t \\
&= -\frac{1}{(-2\pi is)^2}\int_{2\pi is}^{0} ue^u\,\mathrm{d}u + \frac{1}{(-2\pi is)^2}\int_{0}^{-2\pi is} ue^u\,\mathrm{d}u \quad (\text{using } u = -2\pi ist) \\
&= \frac{1}{4\pi^2 s^2}\left[ue^u - e^u\right]_{2\pi is}^{0} - \frac{1}{4\pi^2 s^2}\left[ue^u - e^u\right]_{0}^{-2\pi is} \\
&= \frac{1}{4\pi^2 s^2}\big((0-1) - (2\pi ise^{2\pi is} - e^{2\pi is})\big) - \frac{1}{4\pi^2 s^2}\big((-2\pi ise^{-2\pi is} - e^{-2\pi is}) - (0-1)\big) \\
&= \frac{1}{4\pi^2 s^2}\big((e^{2\pi is} + e^{-2\pi is}) - 2\pi is(e^{2\pi is} - e^{-2\pi is}) - 2\big) \\
&= \frac{1}{4\pi^2 s^2}(2\cos(2\pi s) - 2\pi is \cdot 2i\sin(2\pi s) - 2) \\
&= \frac{1}{4\pi^2 s^2}(2\cos(2\pi s) + 4\pi s\sin(2\pi s) - 2).
\end{aligned}$$

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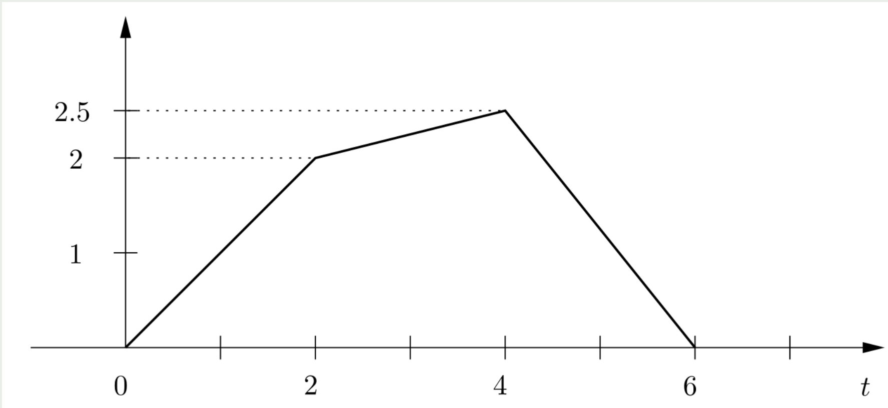
Exercise 2. **Using the shift and stretch properties.**

Find the Fourier transform of the function $f(t)$ whose graph is shown below.



Hint: This graph is the sum of two shifted and stretched triangle functions.

Solution. The function f can be written as a sum of two shifted and stretched triangle functions, namely

$$f(t) = 2\Lambda((t-2)/2) + 2.5\Lambda((t-4)/2).$$

Using the shift and stretch properties of the Fourier transform, we therefore have

$$\begin{aligned}(\mathscr{F}f)(s) &= 2\cdot\mathscr{F}(\Lambda((t-2)/2)) + 2.5\cdot\mathscr{F}(\Lambda((t-4)/2)) \\ &= 2\cdot\mathscr{F}(\Lambda(t/2-1)) + 2.5\cdot\mathscr{F}(\Lambda(t/2-2)) \\ &= 2\cdot 2e^{-4\pi is}(\mathscr{F}\Lambda)(2s) + 2.5\cdot 2e^{-8\pi is}(\mathscr{F}\Lambda)(2s) \\ &= 4e^{-4\pi is}\operatorname{sinc}^2(2s) + 5e^{-8\pi is}\operatorname{sinc}^2(2s) \\ &= \left(4e^{-4\pi is} + 5e^{-8\pi is}\right)\operatorname{sinc}^2(2s).\end{aligned}$$

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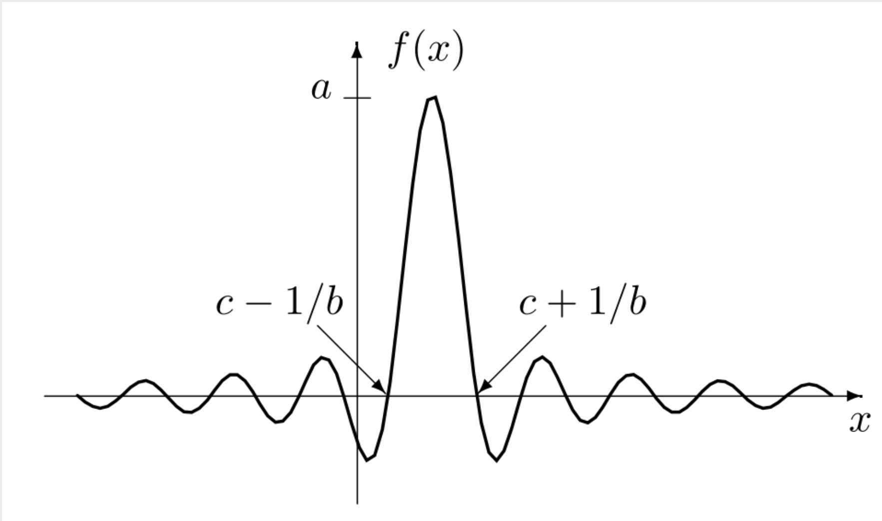
Exercise 3. **Again using the shift and stretch properties.**

Find the Fourier transform of the shifted and scaled sinc function graphed below.



Solution. Observe first that $f(x) = a\,\text{sinc}(b(x-c)) = a\,\text{sinc}(bx-bc)$. By the shifting and stretching properties of the Fourier transform, we then have

$$(\mathscr{F}f)(s) = a\cdot\mathscr{F}(\text{sinc}(bx-bc))$$

$$= a\cdot\frac{1}{|b|}e^{2\pi is(-bc)/b}(\mathscr{F}\,\text{sinc})(s/b)$$

$$= \frac{a}{|b|}e^{-2\pi isc}\Pi(s/b).$$

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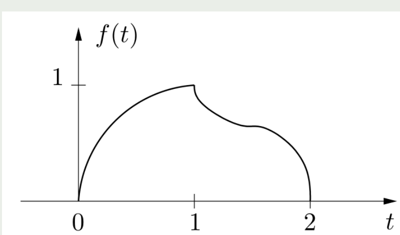
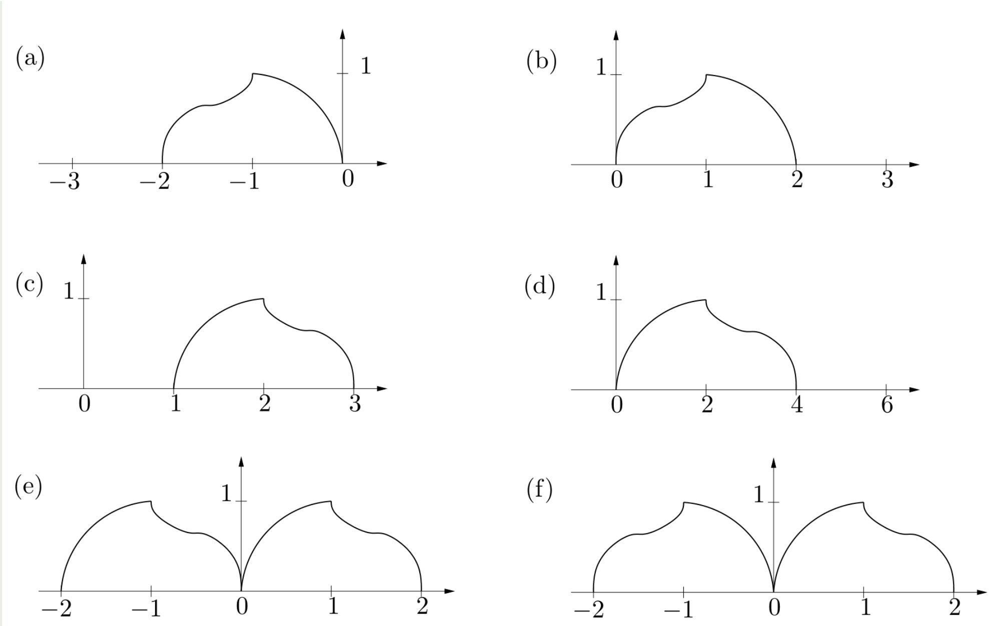
Exercise 4. **Using the shift, stretch, and reverse properties.**

Suppose the function $f(t)$ graphed below has Fourier transform $F(s)$.



Below are six graphs of functions obtained from $f(t)$ by various transformations. Express their Fourier transforms in terms of $F(s)$.



Solution.

a) This function is $f(-t)$, or in terms of the reverse function, f^-. It follows that its Fourier transform is F^-, or equivalently $F(-s)$.

b) This is a reversed and shifted version of f, namely $f^-(t-2)$. You could also write this as $f(-t+2)$. It follows that its Fourier transform is $\frac{1}{|-1|}e^{2\pi is(2)/(-1)}F(s/(-1))$, or equivalently $e^{-4\pi is}F(-s)$.

c) This is a shifted version of f, namely $f(t-1)$. It follows that its Fourier transform is $e^{2\pi is(-1)}F(s)$, or equivalently $e^{-2\pi is}F(s)$.

d) This is a stretched version of f, namely $f(t/2)$. It follows that its Fourier transform is $2F(2s)$.

e) This function is $f(t)+f(t+2)$, so its Fourier transform is $F(s)+e^{4\pi is}F(s)$.

f) This function is $f(t)+f(-t)$, so its Fourier transform is $F(s)+F(-s)$.

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Exercise 5. [Optional] A relationship between Fourier series and the Fourier transform.

Suppose $f(t)$ is a function that is zero outside of $\left(-\frac{1}{2},\frac{1}{2}\right)$. Let $g(t)$ be the function defined below, which agrees with $f(t)$ on $\left(-\frac{1}{2},\frac{1}{2}\right)$ and is periodic with period 1:

$$g(t) = \sum_{n=-\infty}^{\infty} f(t-n).$$

(In Homework 5, we called g a *periodization* of f.) Find a relationship between the Fourier *transform* of f and the Fourier *coefficients* of g.

Solution. Integrating over the cycle $\left[-\frac{1}{2}, \frac{1}{2}\right]$, the Fourier coefficients for g are given by

$$\begin{aligned}
\hat{g}(n) &= \int_{-1/2}^{1/2} g(t) e^{-2\pi i n t}\, \mathrm{d}t \\
&= \int_{-1/2}^{1/2} f(t) e^{-2\pi i n t}\, \mathrm{d}t \quad \text{(since } f \text{ and } g \text{ agree on the interval of integration)} \\
&= \int_{-\infty}^{\infty} f(t) e^{-2\pi i n t}\, \mathrm{d}t \quad \text{(since } f \text{ is zero outside of } [-1/2, 1/2]) \\
&= (\mathscr{F} f)(n).
\end{aligned}$$

So, the Fourier *coefficients* of the periodic function g are exactly the values of the Fourier *transform* of f evaluated at the integers. ■

Additional Properties of the Fourier Transform

Exercise 6. The Fourier transform and differentiation.

This exercise shows how the Fourier transform also behaves well with respect to differentiation, which can make certain computations much easier.

a) Suppose f is differentiable, f' is continuous, and $\lim_{t \to \pm\infty} f(t) = 0$. Show that

$$(\mathscr{F} f')(s) = 2\pi i s (\mathscr{F} f)(s),$$

assuming both transforms exist.

b) Although these hypotheses are not satisfied by the function Λ, use the above property anyway to show $\mathscr{F}\Lambda = \operatorname{sinc}^2$.

c) Suppose $f(t)$ is a solution to the differential equation

$$y''(t) - y(t) + \Lambda(t) = 0.$$

Show that

$$(\mathscr{F} f)(s) = \frac{\operatorname{sinc}^2(s)}{1 + 4\pi^2 s^2}.$$

You may assume whatever is necessary to use the property in (a), e.g., f is smooth and decays rapidly to zero.

Solution.

a) Using integration by parts (which is valid since both f and f' are continuous), we have

$$\begin{aligned}(\mathscr{F}f')(s) &= \int_{-\infty}^{\infty} f'(t)e^{-2\pi ist}\,\mathrm{d}t \\ &= \left[e^{-2\pi ist}f(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)\cdot(-2\pi is)e^{-2\pi ist}\,\mathrm{d}t \\ &= 2\pi is\int_{-\infty}^{\infty} f(t)e^{-2\pi ist}\,\mathrm{d}t \\ &= 2\pi is(\mathscr{F}f)(s).\end{aligned}$$

Note that in the third equality we used the assumption that $\lim_{t\to\pm\infty} f(t) = 0$.

b) First observe that

$$\Lambda'(t) = \begin{cases} 1, & \text{if } -1 < t < 0 \\ -1, & \text{if } 0 < t < 1 \\ 0, & \text{if } |t| > 1 \end{cases}$$

and so $\Lambda'(t) = \Pi(t+1/2) - \Pi(t-1/2)$ (at least for all $t \neq 0, \pm 1$). We therefore have

$$\begin{aligned}(\mathscr{F}\Lambda')(s) &= (\mathscr{F}\Pi(t+1/2))(s) - (\mathscr{F}\Pi(t-1/2))(s) \\ &= e^{\pi is}(\mathscr{F}\Pi)(s) - e^{-\pi is}(\mathscr{F}\Pi)(s) \\ &= \left(e^{\pi is} - e^{-\pi is}\right)(\mathscr{F}\Pi)(s) \\ &= 2i\sin(\pi s)\,\mathrm{sinc}(s).\end{aligned}$$

On the other hand, if we assume the derivative property applies to Λ, then we know $(\mathscr{F}\Lambda')(s) = 2\pi is(\mathscr{F}\Lambda)(s)$, and so we can now conclude

$$\begin{aligned}(\mathscr{F}\Lambda)(s) &= \frac{1}{2\pi is}(\mathscr{F}\Lambda')(s) \\ &= \frac{1}{2\pi is}\cdot 2i\sin(\pi s)\,\mathrm{sinc}(s) \\ &= \frac{\sin(\pi s)}{\pi s}\,\mathrm{sinc}(s) \\ &= \mathrm{sinc}^2(s).\end{aligned}$$

R Why was the derivative property of $\mathscr{F}$ still valid in this case? Let's look at the argument used to prove the derivative property in part (a), in the special case $f(t) = \Lambda(t)$. We will be careful to break each integral at any discontinuities:

$$\begin{aligned}
(\mathscr{F}\Lambda')(s) &= \int_{-\infty}^{\infty} \Lambda'(t)e^{-2\pi ist}\,\mathrm{d}t \\
&= \int_{-1}^{1} \Lambda'(t)e^{-2\pi ist}\,\mathrm{d}t \\
&= \int_{-1}^{0} 1\cdot e^{-2\pi ist}\,\mathrm{d}t + \int_{0}^{1} (-1)\cdot e^{-2\pi ist}\,\mathrm{d}t \\
&= \left[te^{-2\pi ist}\right]_{-1}^{0} - \int_{-1}^{0} t\cdot(-2\pi is)e^{-2\pi ist}\,\mathrm{d}t \\
&\quad + \left[-te^{-2\pi ist}\right]_{0}^{1} - \int_{0}^{1} -t\cdot(-2\pi is)e^{-2\pi ist}\,\mathrm{d}t \\
&= e^{2\pi is} + 2\pi is\int_{-1}^{0} te^{-2\pi ist}\,\mathrm{d}t \\
&\quad - e^{-2\pi is} - 2\pi is\int_{0}^{1} te^{-2\pi ist}\,\mathrm{d}t \\
&= 2\pi is\left(\frac{1}{2\pi is}e^{2\pi is} - \frac{1}{2\pi is}e^{-2\pi is} + \int_{-1}^{0} te^{-2\pi ist}\,\mathrm{d}t - \int_{0}^{1} te^{-2\pi ist}\,\mathrm{d}t\right).
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
2\pi is(\mathscr{F}\Lambda)(s) &= 2\pi is\int_{-\infty}^{\infty} \Lambda(t)e^{-2\pi ist}\,\mathrm{d}t \\
&= 2\pi is\int_{-1}^{1} \Lambda(t)e^{-2\pi ist}\,\mathrm{d}t \\
&= 2\pi is\left(\int_{-1}^{0} (t+1)e^{-2\pi ist}\,\mathrm{d}t + \int_{0}^{1} (1-t)e^{-2\pi ist}\,\mathrm{d}t\right) \\
&= 2\pi is\left(\int_{-1}^{1} e^{-2\pi ist}\,\mathrm{d}t + \int_{-1}^{0} te^{-2\pi ist}\,\mathrm{d}t - \int_{0}^{1} te^{-2\pi ist}\,\mathrm{d}t\right) \\
&= 2\pi is\left(\left[\frac{1}{-2\pi is}e^{-2\pi ist}\right]_{-1}^{1} + \int_{-1}^{0} te^{-2\pi ist}\,\mathrm{d}t - \int_{0}^{1} te^{-2\pi ist}\,\mathrm{d}t\right) \\
&= 2\pi is\left(\frac{1}{2\pi is}e^{2\pi is} - \frac{1}{2\pi is}e^{-2\pi is} + \int_{-1}^{0} te^{-2\pi ist}\,\mathrm{d}t - \int_{0}^{1} te^{-2\pi ist}\,\mathrm{d}t\right).
\end{aligned}$$

So, the property still holds in this case. This is a hint that our requirement that f be differentiable with f' continuous could be relaxed somewhat. I'll leave it to you to ponder what the weakest conditions one needs to impose in order for the property to hold.

c) Applying $\mathscr{F}$ to the equation $f''(t) - f(t) + \Lambda(t) = 0$, we find that

$$(2\pi is)^2(\mathscr{F}f)(s) - (\mathscr{F}f)(s) + \operatorname{sinc}^2(s) = 0.$$

Solving for $\mathscr{F}f$ then yields

$$(\mathscr{F}f)(s) = \frac{\operatorname{sinc}^2(s)}{1+4\pi^2 s^2}.$$

Here we assumed that f was at least twice differentiable, that f' and f'' were continuous, that $\lim_{t\to\pm\infty} f(t) = \lim_{t\to\pm\infty} f'(t) = 0$, and that the Fourier transforms of f, f', and f'' all existed.

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Exercise 7. [Optional] The Fourier transform and multiplication by t.

Verify the following property of the Fourier transform:

$$\mathscr{F}(tf(t)) = \frac{i}{2\pi} \cdot \frac{\mathrm{d}}{\mathrm{d}s}(\mathscr{F}f).$$

Solution. Let's start by computing the derivative on the right-hand side:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}(\mathscr{F}f) &= \frac{\mathrm{d}}{\mathrm{d}s}\int_{-\infty}^{\infty} f(t)e^{-2\pi ist}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial s}\left(f(t)e^{-2\pi ist}\right)\mathrm{d}t \\
&= \int_{-\infty}^{\infty} (-2\pi it)\cdot f(t)e^{-2\pi ist}\,\mathrm{d}t \\
&= -2\pi i\int_{-\infty}^{\infty} tf(t)e^{-2\pi ist}\,\mathrm{d}t \\
&= -2\pi i\cdot\mathscr{F}(tf(t)).
\end{aligned}$$

It follows that

$$\mathscr{F}(tf(t)) = \frac{1}{-2\pi i}\cdot\frac{\mathrm{d}}{\mathrm{d}s}(\mathscr{F}f) = \frac{i}{2\pi}\cdot\frac{\mathrm{d}}{\mathrm{d}s}(\mathscr{F}f).$$

R In words, this property tells us that multiplying a function in the time domain by t corresponds in the frequency domain to taking a derivative (and multiplying by a constant).

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Convolution

Exercise 8. Basic properties of convolution.
Verify the following properties of convolution:

a) $f * g = g * f$

b) $\mathscr{F}^{-1}(f * g) = \left(\mathscr{F}^{-1}f\right)\cdot\left(\mathscr{F}^{-1}g\right)$

c) $\mathscr{F}(fg) = (\mathscr{F}f) * (\mathscr{F}g)$ (*Hint:* See our class notes, or page 98 in Osgood.)

Solution.

a) Starting with the right-hand side and then making a change of variables, we see that

$$\begin{aligned}(g*f)(t) &= \int_{-\infty}^{\infty} g(t-x)f(x)\,\mathrm{d}x \\ &= \int_{\infty}^{-\infty} g(u)f(t-u)\,(-\mathrm{d}u) \quad \text{(using } u=t-x\text{)} \\ &= \int_{-\infty}^{\infty} f(t-u)g(u)\,\mathrm{d}u \\ &= (f*g)(t).\end{aligned}$$

b) We could repeat the calculation we did in class for the Fourier transform, in terms of rewriting the right-hand side as a single integral:

$$\begin{aligned}\left(\mathscr{F}^{-1}f\right)(t)\cdot\left(\mathscr{F}^{-1}g\right)(t) &= \left(\int_{-\infty}^{\infty} f(s)e^{2\pi i s t}\,\mathrm{d}s\right)\left(\int_{-\infty}^{\infty} g(y)e^{2\pi i y t}\,\mathrm{d}y\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)g(y)e^{2\pi i (s+y) t}\,\mathrm{d}s\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u-y)g(y)e^{2\pi i u t}\,\mathrm{d}u\,\mathrm{d}y \quad \text{(using } u=s+y\text{)} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u-y)g(y)e^{2\pi i u t}\,\mathrm{d}y\,\mathrm{d}u \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(u-y)g(y)\,\mathrm{d}y\right)e^{2\pi i u t}\,\mathrm{d}u \\ &= \int_{-\infty}^{\infty} (f*g)(u)e^{2\pi i u t}\,\mathrm{d}u \\ &= (\mathscr{F}^{-1}(f*g))(t).\end{aligned}$$

Alternatively, we could start from the corresponding property of the Fourier transform and then derive the result by taking reverses:

$$\begin{aligned}\mathscr{F}(f*g) = (\mathscr{F}f)\cdot(\mathscr{F}g) \quad &\Leftrightarrow \quad (\mathscr{F}(f*g))^- = (\mathscr{F}f)^-\cdot(\mathscr{F}g)^- \\ &\Leftrightarrow \quad \mathscr{F}^{-1}(f*g) = \left(\mathscr{F}^{-1}f\right)\cdot\left(\mathscr{F}^{-1}g\right).\end{aligned}$$

c) Following our strategy in class, we start by applying $\mathscr{F}$ to the right-hand side of the desired equality:

$$\mathscr{F}\left((\mathscr{F}f)*(\mathscr{F}g)\right) = \mathscr{F}(\mathscr{F}f)\cdot\mathscr{F}(\mathscr{F}g) = f^-\cdot g^- = (fg)^-.$$

Here we used the fact that $\mathscr{F}^2 f = f^-$ for every function, a property we saw in class. Applying $\mathscr{F}$ once more to the equality above gives

$$\mathscr{F}\left(\mathscr{F}\left((\mathscr{F}f)*(\mathscr{F}g)\right)\right) = \mathscr{F}((fg)^-),$$

which again simplifies to

$$((\mathscr{F}f)*(\mathscr{F}g))^- = (\mathscr{F}(fg))^-.$$

Taking reverses on both sides then gives

$$(\mathscr{F}f)*(\mathscr{F}g) = \mathscr{F}(fg).$$

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Exercise 9. [Optional] Convolution and reversal.
If both f and g are reversed, what happens to their convolution, i.e., what is $f^- * g^-$? If only one of f and g is reversed, what happens to their convolution?

Solution. First observe that

$$\begin{aligned}
(f^- * g^-)(t) &= \int_{-\infty}^{\infty} f^-(t-x)g^-(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(x-t)g(-x)\,\mathrm{d}x \\
&= \int_{\infty}^{-\infty} f(-u-t)g(u)\cdot(-\mathrm{d}u) \quad \text{(using } u=-x\text{)} \\
&= \int_{-\infty}^{\infty} f(-t-u)g(u)\,\mathrm{d}u \\
&= (f*g)(-t) \\
&= (f*g)^-(t).
\end{aligned}$$

Thus, $f^- * g^- = (f*g)^-$. In other words, if both f and g are reversed, their convolution is also reversed.

Now let's consider the case when only one of f and g is reversed, say f. Then observe that

$$\begin{aligned}
(f^- * g)(t) &= \int_{-\infty}^{\infty} f^-(t-x)g(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(x-t)g(x)\,\mathrm{d}x \\
&= \int_{\infty}^{-\infty} f(-t-u)g(-u)\cdot(-\mathrm{d}u) \quad \text{(using } u=-x\text{)} \\
&= \int_{-\infty}^{\infty} f(-t-u)g^-(u)\,\mathrm{d}u \\
&= (f*g^-)(-t) \\
&= (f*g^-)^-(t).
\end{aligned}$$

So, in this case we see that $f^- * g = (f*g^-)^-$. We can similarly show that $f*g^- = (f^- * g)^-$. In other words, if you only reverse one of f and g, then their convolution is the reverse of what you would get if you had instead reversed the other function. ∎

Exercise 10. **Computing some self-convolutions.**

a) Show that $\Pi * \Pi = \Lambda$.
Hint: In your integral, separately consider the cases $t \leq -1$, $-1 < t < 0$, $0 \leq t < 1$, and $1 \leq t$. Pictures might help.

b) Let $f(t) = e^{-|t|}$. Determine $f * f$.
Hint: In your integral, separately consider the cases $t \leq 0$ and $t > 0$.

c) Let $g(t) = e^{-\pi t^2}$. Show that $(g * g)(t) = \frac{1}{\sqrt{2}} e^{-\pi t^2/2}$.
Hint: Consider applying $\mathscr{F}$ to $g * g$, simplifying the answer, and then transforming back with $\mathscr{F}^{-1}$.

Solution.

a) By definition, we have

$$\begin{aligned}(\Pi * \Pi)(t) &= \int_{-\infty}^{\infty} \Pi(t-x)\Pi(x)\, \mathrm{d}x \\ &= \int_{-1/2}^{1/2} \Pi(t-x)\, \mathrm{d}x.\end{aligned}$$

Now observe that

$$\Pi(t-x) = \begin{cases} 0, & \text{if } t-x \leq -1/2 \\ 1, & \text{if } -1/2 < t-x < 1/2 \\ 0, & \text{if } t-x \geq 1/2 \end{cases}$$

For $t \leq -1$, we have $t - x \leq -1/2$ for all $x \in [-1/2, 1/2]$. In other words, the square $\Pi(t-x)$ is entirely to the left of the interval $[-1/2, 1/2]$. We therefore have

$$\int_{-1/2}^{1/2} \Pi(t-x)\, \mathrm{d}x = 0.$$

For $-1 < t < 0$, we have $\Pi(t-x) = 1$ for $x \in [-1/2, t+1/2]$ and $\Pi(t-x) = 0$ for $x \in (t+1/2, 1/2]$. In other words, the square $\Pi(t-x)$ partially overlaps the interval $[-1/2, 1/2]$ from the left. We therefore have

$$\int_{-1/2}^{1/2} \Pi(t-x)\, \mathrm{d}x = \int_{-1/2}^{t+1/2} 1\, \mathrm{d}t = t+1.$$

For $0 \leq t < 1$, we have $\Pi(t-x) = 0$ for $x \in [-1/2, t-1/2]$ and $\Pi(t-x) = 1$ for $x \in (t-1/2, 1/2]$. In other words, the square $\Pi(t-x)$ partially overlaps the interval $[-1/2, 1/2]$ from the right. We therefore have

$$\int_{-1/2}^{1/2} \Pi(t-x)\, \mathrm{d}x = \int_{t-1/2}^{1/2} 1\, \mathrm{d}t = 1-t.$$

Finally, for $t \geq 1$, we have $t - x \geq 1/2$ for all $x \in [-1/2, 1/2]$. In other words, the square $\Pi(t-x)$ is entirely to the right of the interval $[-1/2, 1/2]$. We therefore have

$$\int_{-1/2}^{1/2} \Pi(t-x)\,\mathrm{d}x = 0.$$

We've therefore shown that

$$(\Pi * \Pi)(t) = \begin{cases} 0, & \text{if } t \leq -1 \\ t+1, & \text{if } -1 < t < 0 \\ 1-t, & \text{if } 0 \leq t < 1 \\ 0, & \text{if } t \geq 1. \end{cases}$$

Thus, $\Pi * \Pi = \Lambda$.

b) We begin by computing

$$\begin{aligned} (f*f)(t) &= \int_{-\infty}^{\infty} f(t-x) f(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} e^{-|t-x|} e^{-|x|}\,\mathrm{d}x. \end{aligned}$$

Let's first consider the case $t \leq 0$. In that case we have

$$\begin{aligned} (f*f)(t) &= \int_{-\infty}^{t} e^{-|t-x|} e^{-|x|}\,\mathrm{d}x + \int_{t}^{0} e^{-|t-x|} e^{-|x|}\,\mathrm{d}x + \int_{0}^{\infty} e^{-|t-x|} e^{-|x|}\,\mathrm{d}x \\ &= \int_{-\infty}^{t} e^{-(t-x)} e^{x}\,\mathrm{d}x + \int_{t}^{0} e^{t-x} e^{x}\,\mathrm{d}x + \int_{0}^{\infty} e^{t-x} e^{-x}\,\mathrm{d}x \\ &= \int_{-\infty}^{t} e^{-t+2x}\,\mathrm{d}x + \int_{t}^{0} e^{t}\,\mathrm{d}x + \int_{0}^{\infty} e^{t-2x}\,\mathrm{d}x \\ &= \left[\frac{1}{2} e^{-t+2x}\right]_{-\infty}^{t} + \left[e^{t} x\right]_{t}^{0} + \left[-\frac{1}{2} e^{t-2x}\right]_{0}^{\infty} \\ &= \frac{1}{2} e^{t} - t e^{t} + \frac{1}{2} e^{t} \\ &= (1-t) e^{t}. \end{aligned}$$

The analogous computation for $t > 0$ gives

$$(f*f)(t) = (1+t) e^{-t}.$$

Combining these results, we see that

$$(f*f)(t) = (1+|t|) e^{-|t|}.$$

c) Here's a trick for this one. We know that if $g(t) = e^{-\pi t^2}$, then $(\mathscr{F} g)(s) = e^{-\pi s^2}$. Now observe:

$$\mathscr{F}(g*g) = (\mathscr{F} g) \cdot (\mathscr{F} g) = e^{-\pi s^2} \cdot e^{-\pi s^2} = e^{-2\pi s^2}.$$

It follows that

$$\begin{aligned}(g*g)(t) &= \mathscr{F}^{-1}\left(e^{-2\pi s^2}\right) = \mathscr{F}\left(e^{-2\pi s^2}\right)^{-} = \mathscr{F}\left(e^{-2\pi s^2}\right) = \mathscr{F}(g(\sqrt{2}\cdot s)) \\ &= \frac{1}{\sqrt{2}}(\mathscr{F}g)(t/\sqrt{2}) \\ &= \frac{1}{\sqrt{2}}e^{-\pi t^2/2}.\end{aligned}$$

If we had started from the definition, we would have computed

$$(g*g)(t) = \int_{-\infty}^{\infty} e^{-\pi(t-x)^2}\cdot e^{-\pi x^2}\,\mathrm{d}x,$$

which is not a very easy integral to compute.

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Solving Differential Equations with the Fourier Transform

Exercise 11. Solving DEs using the Fourier transform.

For each differential equation below, use the Fourier transform to find a solution $y(t)$ that is a Schwartz function. (Your solution $y(t)$ can be expressed in the form of a convolution integral.)

a) $y'' - y + \Lambda = 0$

b) $y'' - y + \Pi = 0$

c) $y'' - ty = 0$ (*Hint:* Use Exercise 7.)

Solution.

a) Assuming $y(t)$ is a solution that is a Schwartz function, applying $\mathscr{F}$ gives

$$(2\pi i s)^2(\mathscr{F}y)(s) - (\mathscr{F}y)(s) + \operatorname{sinc}^2(s) = 0,$$

which yields

$$(\mathscr{F}y)(s) = \frac{\operatorname{sinc}^2(s)}{1+4\pi^2 s^2} = \frac{1}{1+4\pi^2 s^2}\cdot\operatorname{sinc}^2(s).$$

Applying $\mathscr{F}^{-1}$ then gives

$$\begin{aligned}y(t) &= \mathscr{F}^{-1}\left(\frac{1}{1+4\pi^2 s^2}\right) * \mathscr{F}^{-1}\left(\operatorname{sinc}^2\right) \\ &= \left(\frac{1}{2}e^{-|t|}\right) * \Lambda \\ &= \int_{-\infty}^{\infty}\frac{1}{2}e^{-|t-x|}\Lambda(x)\,\mathrm{d}x \quad \text{(we could stop here)} \\ &= \int_{-1}^{0}\frac{1}{2}e^{-|t-x|}(x+1)\,\mathrm{d}x + \int_{0}^{1}\frac{1}{2}e^{-|t-x|}(1-x)\,\mathrm{d}x.\end{aligned}$$

We could work out these two integrals, but let's stop here.

b) Applying $\mathscr{F}$, we have

$$(2\pi i s)^2(\mathscr{F}y)(s) - (\mathscr{F}y)(s) + \operatorname{sinc}(s) = 0,$$

which yields

$$(\mathscr{F}y)(s) = \frac{\operatorname{sinc}(s)}{1+4\pi^2 s^2} = \frac{1}{1+4\pi^2 s^2} \cdot \operatorname{sinc}(s).$$

Applying $\mathscr{F}^{-1}$ then gives

$$\begin{aligned} y(t) &= \mathscr{F}^{-1}\left(\frac{1}{1+4\pi^2 s^2}\right) * \mathscr{F}^{-1}(\operatorname{sinc}) \\ &= \left(\frac{1}{2}e^{-|t|}\right) * \Pi \\ &= \int_{-\infty}^{\infty} \frac{1}{2}e^{-|t-x|}\Pi(x)\,\mathrm{d}x \quad \text{(we could stop here)} \\ &= \int_{-1/2}^{1/2} \frac{1}{2}e^{-|t-x|}\,\mathrm{d}x. \end{aligned}$$

We could work out this integral, but let's again stop here.

c) Applying $\mathscr{F}$ and using Exercise 7, we have

$$(2\pi i s)^2(\mathscr{F}y)(s) - \frac{i}{2\pi}(\mathscr{F}y)'(s) = 0,$$

which yields

$$(\mathscr{F}y)'(s) = 8i\pi^3 s^2(\mathscr{F}f)(s).$$

Solving this first-order differential equation yields

$$(\mathscr{F}f)(s) = Ae^{\frac{8}{3}i\pi^3 s^3},$$

where A is a constant. Applying $\mathscr{F}^{-1}$, we then have

$$f(t) = \mathscr{F}^{-1}\left(Ae^{\frac{8}{3}i\pi^3 s^3}\right) = A\int_{-\infty}^{\infty} e^{\frac{8}{3}i\pi^3 s^3} \cdot e^{2\pi i s t}\,\mathrm{d}s$$

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