Math 344: Homework 8

Complete by Tuesday, November 27

1. **Directly computing a Fourier transform.**

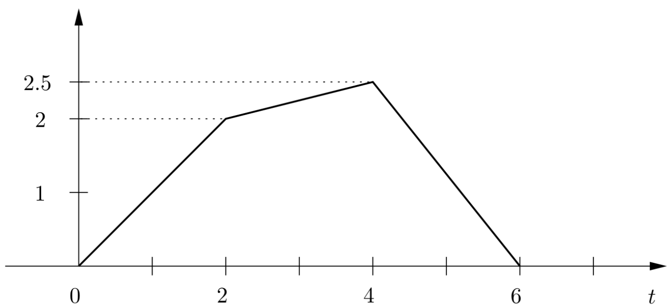
   Consider the function defined below:

$$f(t) = \begin{cases} 0, & \text{if } |t| > 1 \\ -t, & \text{if } -1 \le t < 0 \\ t, & \text{if } 0 \le t \le 1 \end{cases}$$

   This is an even real-valued function, so its Fourier transform is guaranteed to also be an even real-valued function. Compute the Fourier transform of f and write your final answer in the form of a real-valued function, i.e., using only real numbers.

2. **Using the shift and stretch properties.**

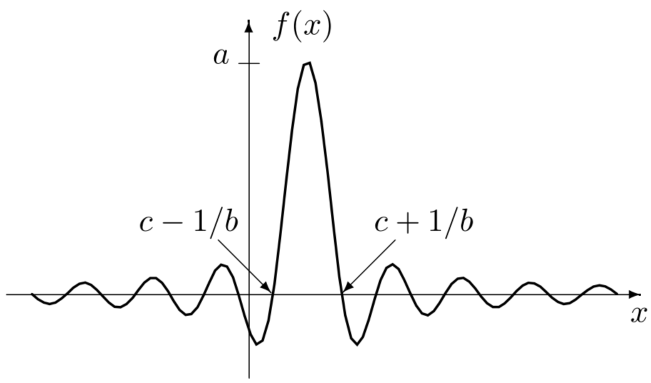
   Find the Fourier transform of the function $f(t)$ whose graph is shown below.



   Hint: This graph is the sum of two shifted and stretched triangle functions.

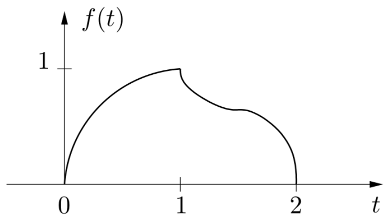
3. **Again using the shift and stretch properties.**

   Find the Fourier transform of the shifted and scaled sinc function graphed below.



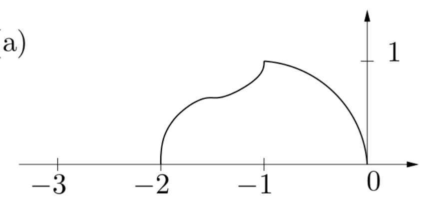
4. **Using the shift, stretch, and reverse properties.**

   Suppose the function $f(t)$ graphed below has Fourier transform $F(s)$.

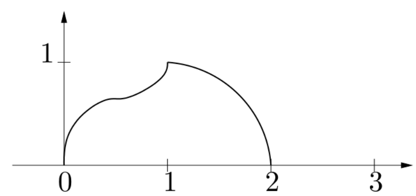


   Below are six graphs of functions obtained from $f(t)$ by various transformations. Express their Fourier transforms in terms of $F(s)$.

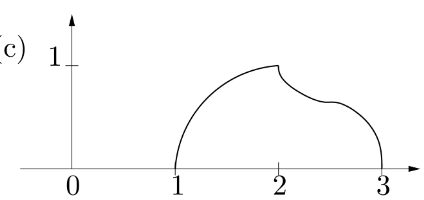
(a)



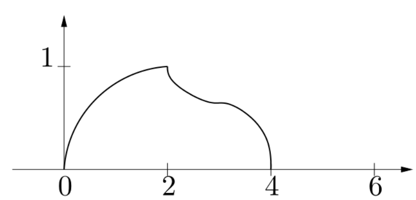
(b)



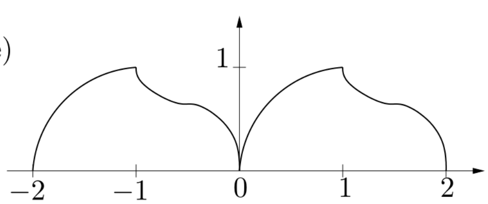
(c)



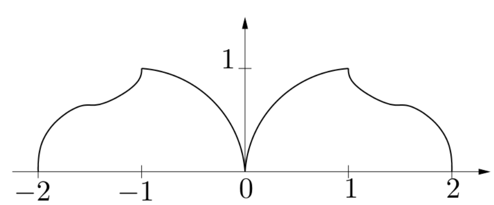
(d)



(e)



(f)



5. **[Optional] A relationship between Fourier series and the Fourier transform.**

   Suppose $f(t)$ is a function that is zero outside of $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Let $g(t)$ be the function defined below, which agrees with $f(t)$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and is periodic with period 1:

$$g(t) = \sum_{n=-\infty}^{\infty} f(t-n).$$

   (In Homework 5, we called g a *periodization* of f.) Find a relationship between the Fourier *transform* of f and the Fourier *coefficients* of g.

6. **The Fourier transform and differentiation.**

   This exercise shows how the Fourier transform also behaves well with respect to differentiation, which can make certain computations much easier.

   a) Suppose f is differentiable, f' is continuous, and $\lim_{t\to\pm\infty} f(t) = 0$. Show that

   $$(\mathscr{F}f')(s) = 2\pi is(\mathscr{F}f)(s),$$

   assuming both transforms exist.

   b) Although these hypotheses are not satisfied by the function Λ, use the above property anyway to show $\mathscr{F}\Lambda = \text{sinc}^2$.

   c) Suppose $f(t)$ is a solution to the differential equation

   $$y''(t) - y(t) + \Lambda(t) = 0.$$

   Show that

   $$(\mathscr{F}f)(s) = \frac{\text{sinc}^2(s)}{1+4\pi^2 s^2}.$$

   You may assume whatever is necessary to use the property in (a), e.g., f is smooth and decays rapidly to zero.

7. **[Optional] The Fourier transform and multiplication by t.**

   Verify the following property of the Fourier transform:

   $$\mathscr{F}(tf(t)) = \frac{i}{2\pi} \cdot \frac{\mathrm{d}}{\mathrm{d}s}(\mathscr{F}f).$$

8. **Basic properties of convolution.**
   Verify the following properties of convolution:

   a) $f * g = g * f$

   b) $\mathscr{F}^{-1}(f * g) = \left(\mathscr{F}^{-1}f\right) \cdot \left(\mathscr{F}^{-1}g\right)$

   c) $\mathscr{F}(fg) = (\mathscr{F}f) * (\mathscr{F}g)$ (*Hint:* See our class notes, or page 98 in Osgood.)

9. **[Optional] Convolution and reversal.**
   If both f and g are reversed, what happens to their convolution, i.e., what is $f^- * g^-$? If only one of f and g is reversed, what happens to their convolution?

10. **Computing some self-convolutions.**

a) Show that $\Pi * \Pi = \Lambda$.
Hint: In your integral, separately consider the cases $t \leq -1$, $-1 < t < 0$, $0 \leq t < 1$, and $1 \leq t$. Pictures might help.

b) Let $f(t) = e^{-|t|}$. Determine $f * f$.
Hint: In your integral, separately consider the cases $t \leq 0$ and $t > 0$.

c) Let $g(t) = e^{-\pi t^2}$. Show that $(g * g)(t) = \frac{1}{\sqrt{2}} e^{-\pi t^2/2}$.
Hint: Consider applying $\mathscr{F}$ to $g * g$, simplifying the answer, and then transforming back with $\mathscr{F}^{-1}$.

11. **Solving DEs using the Fourier transform.**

For each differential equation below, use the Fourier transform to find a solution $y(t)$ that is a Schwartz function. (Your solution $y(t)$ can be expressed in the form of a convolution integral.)

a) $y'' - y + \Lambda = 0$

b) $y'' - y + \Pi = 0$

c) $y'' - ty = 0$ (*Hint:* Use Exercise 7.)