Math 344: Quiz 6 Solutions

Fall 2018

Problem 1. Compute the Fourier transform of the function f defined below:

$$f(t) = \begin{cases} 0, & \text{if } |t| > 1 \\ -t, & \text{if } -1 \leq t < 0 \\ t, & \text{if } 0 \leq t \leq 1 \end{cases}$$

(*Hint:* You may use the antiderivative formula $\int ue^u \, \mathrm{d}u = ue^u - e^u + C$.)

Solution. Using our integral definition of the Fourier transform, we compute

$$\begin{aligned}
(\mathscr{F} f)(s) &= \int_{-\infty}^{\infty} f(t) e^{-2\pi its} \, \mathrm{d}t \\
&= \int_{-1}^{0} -te^{-2\pi its} \, \mathrm{d}t + \int_{0}^{1} te^{-2\pi its} \, \mathrm{d}t \\
&= -\frac{1}{(-2\pi is)^2} \int_{2\pi is}^{0} ue^u \, \mathrm{d}u + \frac{1}{(-2\pi is)^2} \int_{0}^{-2\pi is} ue^u \, \mathrm{d}u \quad (\text{using } u = -2\pi ist) \\
&= \frac{1}{4\pi^2 s^2} \left[ue^u - e^u\right]_{2\pi is}^{0} - \frac{1}{4\pi^2 s^2} \left[ue^u - e^u\right]_{0}^{-2\pi is} \\
&= \frac{1}{4\pi^2 s^2} \left((0-1) - (2\pi ise^{2\pi is} - e^{2\pi is})\right) - \frac{1}{4\pi^2 s^2} \left((-2\pi ise^{-2\pi is} - e^{-2\pi is}) - (0-1)\right) \\
&= \frac{1}{4\pi^2 s^2} \left((e^{2\pi is} + e^{-2\pi is}) - 2\pi is(e^{2\pi is} - e^{-2\pi is}) - 2\right) \\
&= \frac{1}{4\pi^2 s^2} (2\cos(2\pi s) + 4\pi s \sin(2\pi s) - 2) \quad (\text{optional simplification}).
\end{aligned}$$

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Problem 2. Suppose $f(t)$ is a solution to the differential equation

$$y''(t) - y(t) + \Pi(t) = 0.$$

Determine $\mathscr{F} f$. You may assume f is smooth and rapidly decays to zero, so that the $\mathscr{F} f$ exists and all of the nice properties of $\mathscr{F}$ hold.

Solution. Applying $\mathscr{F}$ to the equation $f''(t) - f(t) + \Pi(t) = 0$, we find that

$$(2\pi is)^2 (\mathscr{F} f)(s) - (\mathscr{F} f)(s) + \operatorname{sinc}(s) = 0.$$

Solving for $\mathscr{F} f$ then yields

$$(\mathscr{F} f)(s) = \frac{\operatorname{sinc}(s)}{1 + 4\pi^2 s^2}.$$

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