Math 344: Quiz 7 Solutions

Fall 2018

Problem Use the Laplace transform to solve the initial-value problem

$$y' + 2y = 2u_1(t), \quad y(0) = 1.$$

Hint: The following equality may be useful: $\frac{2}{s(s+2)} = \frac{1}{s} - \frac{1}{s+2}$.

Solution. Applying the Laplace transform to the original differential equation produces an algebraic equation, which we can then solve for Ly:

$$\begin{aligned}
Ly' + 2\cdot Ly = 2\cdot L(u_1(t)) &\Leftrightarrow (s\cdot Ly - y(0)) + 2\cdot Ly = 2\cdot e^{-s}\cdot\frac{1}{s} \\
&\Leftrightarrow (s\cdot Ly - 1) + 2\cdot Ly = e^{-s}\cdot\frac{2}{s} \\
&\Leftrightarrow (s+2)\cdot Ly = e^{-s}\cdot\frac{2}{s} + 1 \\
&\Leftrightarrow Ly = e^{-s}\cdot\frac{2}{s(s+2)} + \frac{1}{s+2} \\
&\Leftrightarrow Ly = e^{-s}\cdot\left(\frac{1}{s} - \frac{1}{s+2}\right) + \frac{1}{s+2}.
\end{aligned}$$

Note that we used partial fractions to obtain the last equality. Applying the inverse transform then gives

$$\begin{aligned}
y &= L^{-1}\left(e^{-s}\cdot\left(\frac{1}{s} - \frac{1}{s+2}\right)\right) + L^{-1}\left(\frac{1}{s+2}\right) \\
&= u_1(t)\cdot\left(L^{-1}\left(\frac{1}{s} - \frac{1}{s+2}\right)\right)\Big|_{t\mapsto t-1} + e^{-2t} \\
&= u_1(t)\cdot\left(1 - e^{-2t}\right)\big|_{t\mapsto t-1} + e^{-2t} \\
&= u_1(t)\cdot\left(1 - e^{-2(t-1)}\right) + e^{-2t}.
\end{aligned}$$

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Laplace Transform Identities

$$L(t^n) = \tfrac{n!}{s^{n+1}}, \qquad L(e^{at}) = \tfrac{1}{s-a}, \qquad L(\sin(bt)) = \tfrac{b}{s^2+b^2}, \qquad L(\cos(bt)) = \tfrac{s}{s^2+b^2}$$

$$L(u_b(t)f(t)) = e^{-bs}\cdot L(f(t+b)), \qquad L^{-1}(F(s+b)) = e^{-bt}\cdot L^{-1}F$$

$$Lf^{(n)} = s^n\cdot Lf - s^{n-1}\cdot f(0) - s^{n-2}\cdot f'(0) - \cdots - f^{(n-1)}(0)$$